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A REDUNDANCY NOTEBOOK

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The objective of the report is to present in a coherent fashion the information and tools necessary for the evaluation of most types of redundancy design configurations with which a reliability engineer is faced. The report contains a number of alternative evaluation approaches, both classical and unique. Closed form results and algorithms are derived for the evaluation of the reliability of various types of redundant configurations.

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CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 The subject of redundancy has been a popular topic for papers on reliability since the late 1950's. Most probably Von Neumann's contributions in the 1940's relative to the application of Majority Voting schemes to computers provided the nucleus for much of this work (although his original efforts were not concerned primarily with improving electronic hardware reliability). Although many authors wrote on the more simplistic features of redundancy, three stand out as those who are referenced frequently in papers by other authors: Balaban (4)*, Moskowitz (28), and Kneale (25). They wrote primarily about unattended redundant systems, for the most part ignoring the effect of sensing and switching elements. The impact of such factors was considered later by such people as Grisamore (11) and Aroian (2). In all of these, the figures of merit of concern were Reliability, expressed as the probability that the systems would remain operational over a given period of time, and Mean Time to First System Failure.

In the early 1960's, interest began to center on formulations for redundant systems which were maintained. At this time, the traditional reliability figures of merit were augmented by an additional figure of merit called Availability. During this era the primary tool used was the Markovian process. The foremost pioneer investigators in the area of maintained systems were Barlow and Hunter (6), who introduced what is now known as the Availability measures (both time dependent and limiting) for full on systems. Shortly thereafter, Epstein and Hosford (17) for the first time defined the reliability, probability of no system failure over a period of operating time, and mean time to first failure of both full and standby redundant configurations. Later Dick (12) defined the Availability figures of merit for a group of two unit redundant systems under different operational scenarios.

The Markovian Procedure was in general rather cumbersome to use and as a consequence, in the years that followed, approximation procedures were a topic of quite a few papers; for example, McGregor (27), Applebaum (1), and Einhorn (16). Dick (13) was the first to develop a rather simple approach to evaluate the mean time to first failure of a full on redundant system.

The purpose of this report is twofold:

(a) to present the information and tools necessary to evaluate most of the types of redundancy problems with which a reliability engineer is faced;

(b) to present new simplified approaches to redundancy analyses which provide time savings compared to the classical methods.

As indicated previously, many papers have been published about redundancy. Most are repetitious, except for minor variations. Those that are not repetitious are published in various reports or symposium proceedings, widely separated by years.

To the typical analyst charged with the evaluation of the reliability of a redundant method, this presents serious problems. What is required, and part of the subject to which this document pertains, is a survey of the basic redundancy literature to make available in a single document, information such that the vast majority of redundancy design applications encountered can be evaluated.

For the most part, the most widely used and strongest evaluation techniques available are complex and time-consuming to apply (especially for repairable redundant networks). This document contains unique evaluation approaches and/or results which are in many instances less complex and less time-consuming than the traditional approaches.

A summary of the subjects covered in this thesis is shown in Table 1.1.

1.2 Background

In order to cope with the military technological developments of the past fifteen years, electronic systems have been compelled to expand in both size and complexity at a rapid rate. Of equivalent importance to the need for this growth has been the coincidental need for greater system reliability and maintainability. As military and space requirements necessitate the construction of even more complex systems, the contemporary philosophies of reliability and maintainability will become inadequate for the successful performance of mission objectives. The most severely affected systems will be those on which maintenance cannot be performed, such as satellite-borne systems, and

TABLE 1.1

A SUMMARY OF REDUNDANCY TOPICS

NON-REPAIRABLE SYSTEMS	REPAIRABLE SYSTEMS
Full on Redundancy • **Traditional Approach** • **State Analysis Approach**	Full on Redundancy • Markovian Approach • Combined Unit Approach • Expectation/Transition Approach • System Failure Rate Approach • Periodically Maintained Systems • Impact of Redundancy on Maintainability
Standby Redundancy • **Traditional Approach** • **Perfect Switching** • **Imperfect Switching**	
Efficient Levels of Redundancy	Standby Redundancy • Markovian Approach • Expectation Transition Model

strategic military systems where the luxury of even a small down time cannot be afforded. At this time, the only recourse to such situations is the creation of components of increased reliability or the application of redundancy.

In order to increase the reliability of complex electronic systems, a constant effort is made to improve the reliability of the component parts comprising such systems. The rate of improvement of component part reliability, however, lies significantly below the rate of increase of system complexity. In addition, it must be realized that the attainment of 100% reliability for a component part is impossible. The only recourse then is redundancy (the addition of duplicate elements).

Redundancy may be achieved in many ways. Each has its advantages (reliability gain) and its disadvantages (the number of duplicative elements required which impact on total system weight, cost, volume). The purpose of this report is to explain the rationale for each type of redundancy considered and to develop means of evaluation such that the reliability potential of each may be assessed and tradeoffs made.

1.2.1 Measures of Reliability

The following reliability measures will be used in this study. (1) Mean Time to Failure M; (2) Probability of Failure free operation for specified time t, denoted by R(t); (3) P(t), the probability that a system will be functioning at time t. For the non-maintained system $R(t) = P(t)$ and for the maintained system $R(t) \neq P(t)$.

The following relationships exist:

$$M = \int_0^{\infty} t\, W(t)\,dt \tag{1.1}$$

where W(t) denotes the failure density function.
Now:

$$W(t) = -\frac{dR(t)}{dt} \tag{1.2}$$

Hence:

$$M = \int_0^{\infty} - t \frac{dR(t)}{dt} \, dt \tag{1.3}$$

Integrating by parts:

$$M = - t R(t) \Big|_0^{\infty} + \int_0^{\infty} R(t) dt \tag{1.4}$$

Now $O \cdot R(O) = 0$ and

$$\lim_{t \to \infty} t R(t) = \lim_{t \to \infty} t e^{-\int_0^t h(x) dx} \to 0$$

where h(x) denotes the hazard rate, i.e. $h(x) = \frac{W(x)}{R(x)}$.
Hence:

$$M = \int_0^{\infty} R(t) dt \tag{1.5}$$

1.2.2 Definition of R(t):

Since the expression for R(t) is a probabilistic function, its formulation will involve the combination of probabilities (Reliabilities) of success, or survival (over a given period of time), for all the units making up the system in question. Where redundancy does not exist, the failure of any one unit results in system failure and R(t) is comprised of the product of the reliabilities (probabilities of survival for a given period of time) of all units comprising the system.

$$R(t) = \prod_{i=1}^{N} R(t)_i \tag{1.6}$$

When the system is composed of redundant units, numerous possibilities for system survival exist (with no redundancy the system can survive only if no units fail in the given period of time), hence R(t) must be defined in terms of complex combinations of probabilities rather than as a simple product. For this reason, the following section on simple probability theory will serve as necessary background for the definition of R(t).

1.2.3 Probability Basics

Given two mutually exclusive events A and B, the probability of either occurring is the sum of their probabilities:

$$P(A+B) = P(A) + P(B)$$

This follows from the fact that, in general, if we have K mutually exclusive events $B_1, B_2, B_3, \ldots B_k$, then:

$$P(B_1+B_2..B_k) = \sum_{i=1}^{K} P(B_i)$$

If two events exist that are not mutually exclusive (say A and B), the probability of one or more occurring is:

$$P(A+B) = P(A) + P(B) - P(AB)$$

For three such events:

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AC) - P(AB) - P(BC)$$

$$+ P(ABC)$$

Generalizing to K non-mutually exclusive events $B_1, B_2 \ldots B_K$, the probability of one or more of the events occurring may be defined as:

$$P(B_1 + B_2 \ldots B\ ) = \sum_{i}^{K} P(B_i) - P(B_1B_2)$$

$$- P(B_1B_3) - \ldots . P(B_{K-1}B_K) + P(B_1B_2B_3)$$

$$+ P(B_1B_2B_4) + P(B_{K-2}B_{K-1}B_K) + \ldots . .$$

$$+(-1)^{K-1} P(B_1B_2 \ldots B_K) \ldots . .$$

If events $A_1, A_2, A_3 \ldots A_K$ are independent and the probability of occurrence of all events is desired, the probability that all occur $P(A_1A_2A_3 \ldots A_K)$ may be calculated as:

$$P(A_1A_2 \ldots A_K) = \prod_{i=1}^{K} P(A_i)$$

Equating the term, event, with either a failure or a success and the probability of an event with the probability of failure or the probability of success, the relationship between R(t) and probability theory is immediately evident.

1.2.4 The Block Diagram

In order to complete the picture for evaluation of R(t), the concepts of probability must be applied to the system block diagram. From the reliability block diagram, and the definition of each block's reliability (probability of survival for a given period of time) an expression defining system reliability may be developed.

Figure 1.1 shows such a block diagram made up of two units, A and B. The system will operate successfully if either unit A or unit B or both units are operative and will be considered as failed if both A and B fail.

Let us consider the event that A survives a period of operation and define the probability of such an event as $R(t_A)$.

Let us consider the event that B survives a period of operation and define the probability of such an event as $R(t_B)$.

Note that the survival or failure of either A or B does not and will not affect the survival or failure of the other, hence A and B are considered independent.

Note also that just because unit A survives, unit B does not have to fail and vice versa (both units can fail or survive); hence the events are not mutually exclusive.

Therefore, the probability of system survival is equal to the probability that A or B or both survive.

$$P(A+B) = P(A) + P(B) - P(AB)$$

or

$$R(t) = R(t_A) + R(t_B) - R(t_A) \cdot R(t_B)$$

Figure 1.2 shows a block diagram made up of three units: A,B, and C in series. The system will operate successfully only if all three units are operative. The system will fail as soon as one of the three fails.

Let us consider the events that A, B, and C survive a period of operation and define respectively the probability of such an event for each $R(t_A)$, $R(t_B)$,. $R(t_C)$.

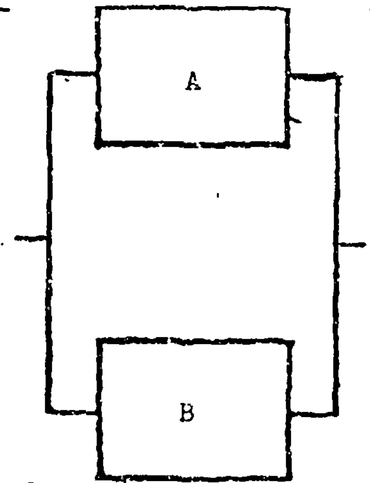


Figure 1.1 Two Redundant Units

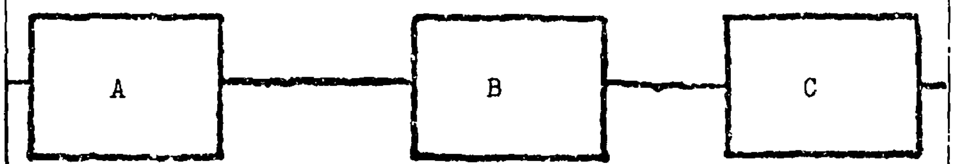


Figure 1.2 Three Series Units

$$P(A, B, C) = P(A) \cdot P(B) \cdot P(C)$$

or

$$R(t) = R(t_A) \cdot R(t_B) \cdot R(t_C)$$

1.2.5 Types of Redundancy:

The discussions that follow will concern themselves with two basic classes of Redundant Systems:

A. Systems which are redundant and which are non-maintained (failed units of a redundant complex are not repaired or replaced). This situation is common to unattended applications, i.e., an unmanned field site, a satellite etc.

B. Systems which are redundant and maintained (failed units of a redundant complex are repaired and replaced). This situation is common to attended applications, i.e., manned site.

For each class several types or varieties of redundancy are considered. In all cases we will assume that the units have times to failure having an exponential distribution described by the probability density function:

$$r(t,\lambda) = \lambda e^{-\lambda t} \qquad \begin{matrix} t > 0 \\ \lambda > 0 \end{matrix}$$

λ = failure rate of unit

t = operating time in question, and

$$R(t) = \int_t^{\infty} r(t,\lambda)\,dt = e^{-\lambda t} = \text{probability that unit will not fail during operating time } t.$$

CHAPTER 2

RELIABILITY MODELS FOR NON-REPAIRABLE SYSTEMS

2.1. Full on Redundancy (Single Unit Necessary For Survival)

The most widely discussed form of redundancy for a situation not involving repair has been a series arrangement of a number of (N) redundant elements as shown in Figure 2.1. In this type of redundancy, all L•N units are continuously energized,* and it is assumed that so long as at least one unit is functioning properly on each of the L cascaded subsystems, the system as a whole will operate successfully. The reliability of such a redundant system is obtained as follows:

Let: L= number of cascaded subsystems composing the system.

N= number of continuously energized units comprising each subsystem.

λ= failure rate of each redundant unit.

(For convenience, N and λ are taken to be the same for all units comprising the system).

* - Hence the term Full on Redundancy.

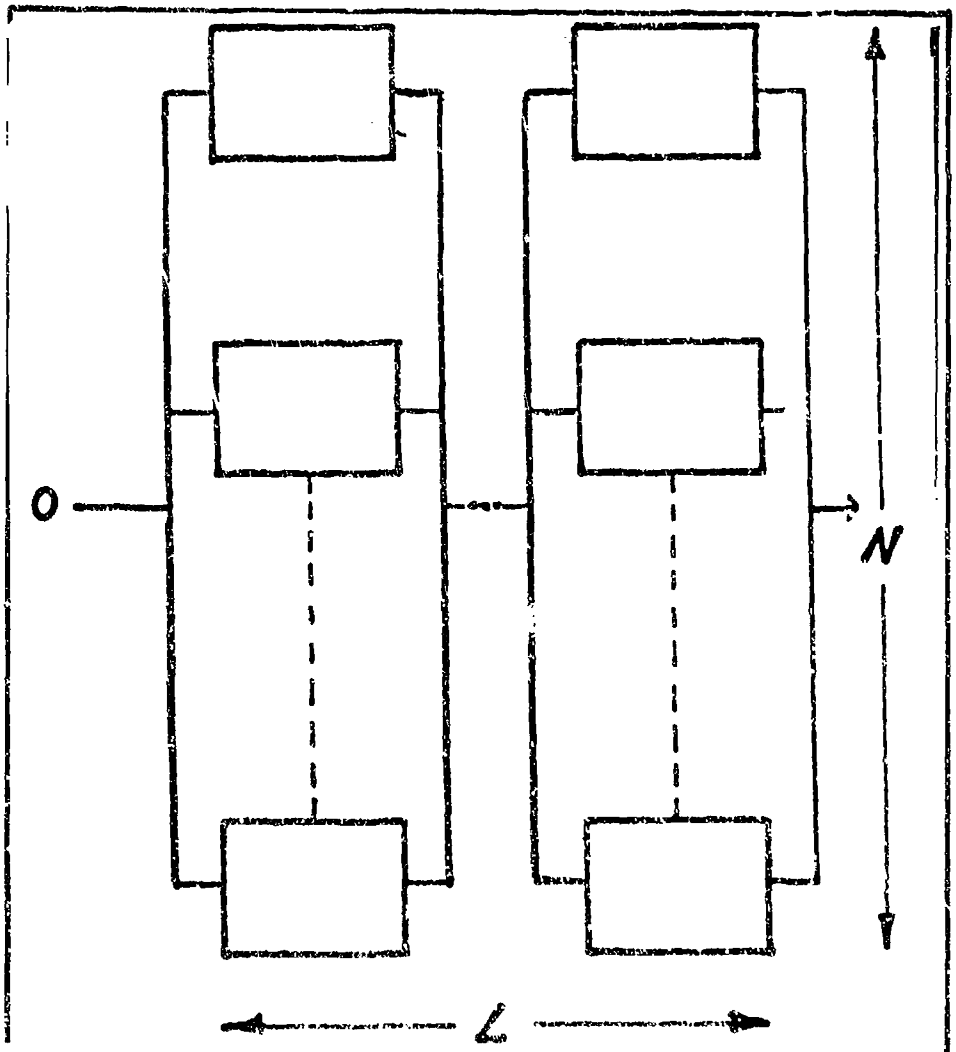


Figure 2.1. L Redundant Subsystems in Series

We assume that the failure rate of each unit is given by an exponential distribution with parameter λ. Then we have:

— The reliability of any unit (i) equal $R(t) = e^{-\lambda t}$.
— [The probability of any unit failing in time t] $= 1-e^{-\lambda t}$.
— [The probability of all N units in a subsystem failing in time t] $= (1-e^{-\lambda t})^N$.
— The probability of at least one unit in a given subsystem surviving $= [1-(1-e^{-\lambda t})^N]$.

The probability of all L subsystems having at least one operating unit is thus given by:

$$R(t) = [1-(1-e^{-\lambda t})^N]^L \tag{2.1}$$

As shown earlier, the mean time to failure is $\int_0^{\infty} R(t)dt$. Therefore, the system mean time to first failure is:

$$M = \int_0^{\infty} \{1-(1-e^{-\lambda t})^N\}^L dt \tag{2.2}$$

After some manipulations, (see Appendix A), this expression reduces to the following simple form:

$$M = 1/\lambda \sum_{k=1}^{L} (-1)^{k+1} \binom{L}{k} \sum_{s=1}^{kN} 1/s \tag{2.3}$$

The quantity of interest, λM, where:

$$\lambda M = \frac{\text{MTBF of the system}}{\text{MTBF of a unit}}$$

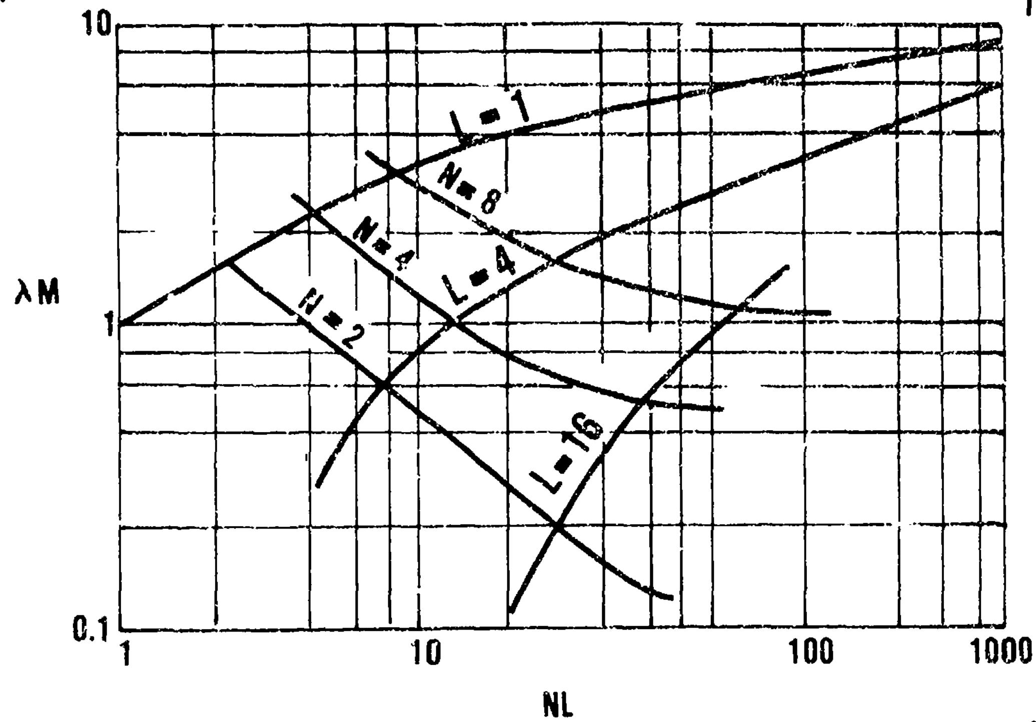
is thus given by:

$$\lambda M = \sum_{k=1}^{L} (-1)^{k+1} \binom{L}{k} \sum_{s=1}^{kN} 1/s \tag{2.4}$$

The mean times to failure of redundant systems which selected values of N and L are shown in Figure 2.2. As can be noted from this figure, order of magnitude increases in system mean life over element mean life are not possible unless extremely large values of N are employed.

A special case is considered when L= 1. This defines a single subsystem composed of N parallel (redundant) units. It follows from Equation (2.1) that the relationship depicting reliability is:

$$R(t) = 1-(1-e^{-\lambda t})^N \tag{2.5}$$



NL = Total number of components in system. A factor of improvement of redundant system mean life over simplex system mean life may be found by multiplying the value determined for λM by L. λ = failure rate of a single system element: M = system mean life

Figure 2.2. λM Relationships For Full On Systems

with corresponding meantime to first failure:

$$M = \int_0^\infty \left[1 - (1-e^{-\lambda t})^N\right] dt$$

$$= 1/\lambda \sum_{s=2}^{N} 1/s = 1/\lambda + 1/2\lambda + 1/3\lambda + \ldots 1/N\lambda \quad (2.6)$$

Note that when $N=2$, the familiar 2 unit redundancy example, we have:

$$M=\frac{3}{2\lambda}$$

The above equations provide a means to develop the R(t) relationship for a system containing L subsystems each with a different number of units N in parallel. Thus:

$$R(t)=\prod_{i=1}^{L}\left[1-(1-e^{-\lambda t})^{N_i}\right] \tag{2.7}$$

with corresponding mean time to first failure given by:

$$M=\int_0^{\infty}\prod_{i=1}^{L}\left[1-(1-e^{-\lambda t})^{N_i}\right]dt \tag{2.8}$$

2.2 Binomial Redundancy (multiple operating units required); Full on Redundancy

This type of redundancy is identical to the one described above except that the system may be considered as a large subsystem composed of N fully energized parallel units and requires a minimum of $D(D\leq N)$ operating units (non-failed units) in order to operate.

This type of a system may be described probabilistically by the binomial distribution:

$$R(t)=\sum_{k=D}^{N}\binom{N}{k}(e^{-\lambda t})^{k}(1-e^{-\lambda t})^{N-k} \tag{2.9}$$

with corresponding mean time to first failure given by:

$$M=\int_0^{\infty}R(t)dt=1/\lambda\sum_{k=D}^{N}1/k \tag{2.10}$$

2.3 State Transition Model: Full on Redundancy

An alternate means of developing the relationship for mean time to first failure is to consider the concept of system states and the concept of transition from one state to another.

The system starts out initially with N units operating. Since we are not considering the concept of repair in this particular situation, the system will experience a unit failure and be reduced to (N-1) operating units; will eventually experience a second unit failure and be reduced to (N-2) operating units, etc., until only D units are operating. The next failure which occurs results in only (D-1) units operating and will cause the system to fail.

Let E_{N-k} $(k=0,1,\ldots N-D+1)$ represent the state that the system has (N-k) units operating. Since this is a non-repairable system, the system will transition from state E_{N-k} to state E_{N-k-1}. Each state has an average time to transition into the next possible state (E_{N-k} to E_{N-k-1}) given by:

$$M_{(N-k/N-k-1)} = 1/\lambda(N-k)$$

λ= failure rate of each unit.

Since the transitions must occur in sequence, the mean time to transition from state E_N to E_{D-1} is given by:

$$M = \sum_{k=0}^{N-D} M_{(N-k/N-k-1)} = \sum_{k=0}^{N-D} 1/\lambda(N-k)$$

$$= 1/\lambda \sum_{k=D}^{N} 1/k$$

and which corresponds to equation (2.10).

Normalizing as before yields:

$$\lambda M = \sum_{k=D}^{N} 1/k \qquad (2.11)$$

Figure 2.3 shows a plot of reliability improvement for this type of redundancy.

2.4 Standby Redundancy (single unit necessary for survival)

A second type of redundant system which possesses a similar configuration to the system previously described, but employs switched-in redundancy is that illustrated in Figure 2.4.

In this instance only one element of each of the cascaded subsystems is activated at a time. Upon failure of this element, the next element of the subsystem will automatically be switched into operation. Until such a switch occurs, the standby element is not energized and hence a failure rate $\lambda = 0$ is assumed. In order to hypothesize an upper bound for the reliability of such a system, the failure rate of the switching device will be equated to 0.

In this model, the successive failures form a Poisson process with rate λ. Then the reliability of the system will be given by:

$$R(t) = \sum_{r=0}^{N-1} P(r,t)$$

$$= \sum_{r=0}^{N-1} e^{-\lambda t} \frac{(\lambda t)^r}{r!} \qquad (2.12)$$

where $P(r,t)$ = Probability of r failures in time t and λt represents the number of failures expected in the time period t, and N the number of redundant units composing each subsystem.

Since:

$$\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^r}{r!} dt = 1/\lambda$$

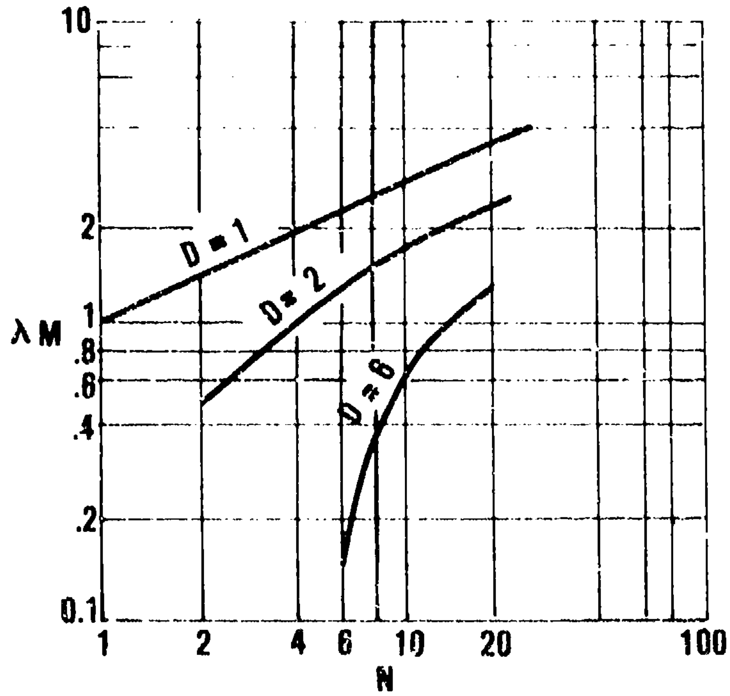
we have the mean time to first failure as before,

$$M = \int_0^\infty R(t)\,dt$$

$$= \int_0^\infty \sum_{r=0}^{N-1} e^{-\lambda t} \frac{(\lambda t)^r}{r!}\,dt \quad .$$

$$= \sum_{r=0}^{N-1} 1/\lambda$$

$$= N/\lambda \qquad (2.13)$$



N = number of units in a subsystem
λ = failure rate of any active subsystem unit
D = number of units necessary for subsystem operation

Figure 2.3 λM Relationships for Full on Subsystems

and

$$\lambda M = N. \tag{2.14}$$

The reliability of a system composed of L such subsystems may be shown to be:

$$R(t) = \left[e^{-\lambda t} \sum_{r=0}^{N-1} \frac{(\lambda t)^r}{r!}\right]^L \tag{2.15}$$

The mean time to first failure of such a system is given by:

$$M = \int_0^{\infty} R(t)\,dt = \int_0^{\infty} \left[e^{-\lambda t} \sum_{r=0}^{N-1} \frac{(\lambda t)^r}{r!}\right]^L dt \tag{2.16}$$

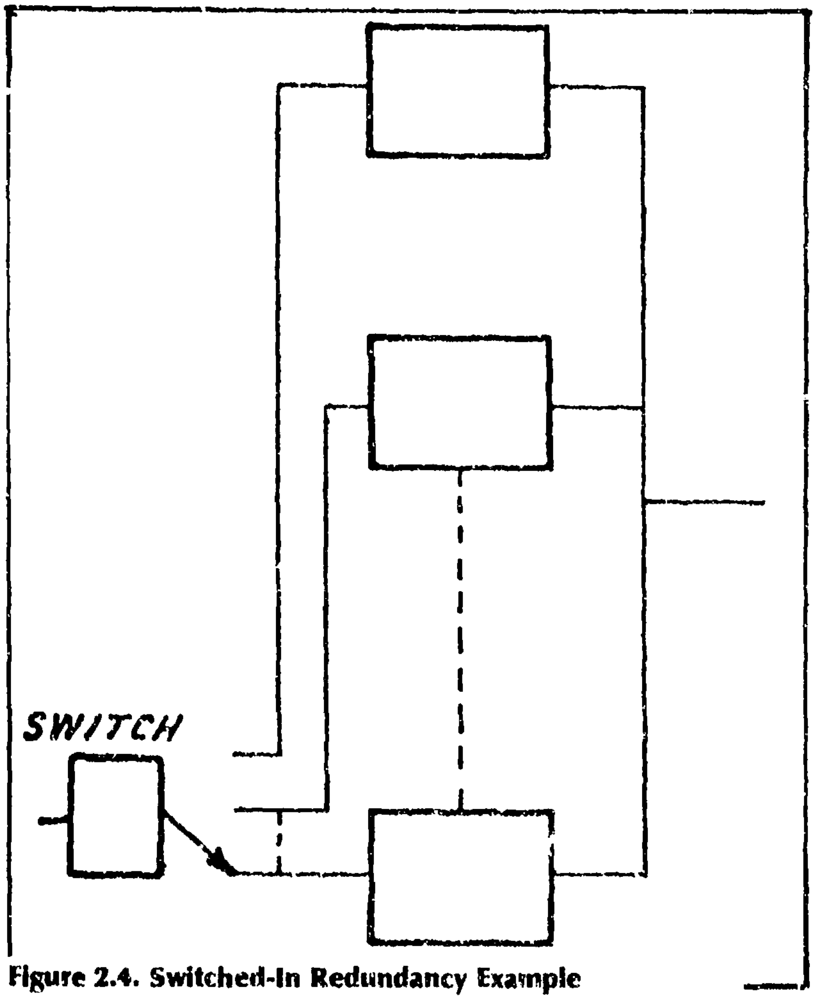


Figure 2.4. Switched-In Redundancy Example

A general evaluation of this equation for various values of L and N is difficult. However, if a specific value of N is stipulated, an explicit form can be obtained for any value of L by a series expansion.

Let N= 2, then:

$$M = \int_0^\infty \left[e^{-\lambda t} \sum_{r=0}^{N-1} \frac{(\lambda t)^r}{r!}\right]^L dt$$

$$= \int_0^\infty e^{-L\lambda t} \left(\sum_{r=0}^{1} \frac{(\lambda t)^r}{r!}\right)^L dt$$

$$= \int_0^\infty e^{-L\lambda t} (1 + \lambda t)^L \, dt$$

$$= \int_0^\infty e^{-L\lambda t} \sum_{r=0}^{L} \binom{L}{r} (\lambda t)^r \, dt$$

$$= \sum_{r=0}^{L} \frac{L!}{\lambda L^{(r+1)} \, (L-r)!} \qquad (2.17)$$

where the last equation is obtained by integration by parts.

A plot of improvement of N= 2 appears in Figure 2.5. In order to more easily evaluate the orders of improvement in mean life reali; ·d in this type of redundancy, a comparison on a subsystem level is made in Figure 2.6 of this concept of redundancy, vs. the concept depicted in Figure 2.3.

As may be noted from Figure 2.6, greater increases in magnitude in system mean life over element mean life may be realized by utilizing this concept rather than the full on redundancy concept previously described.

Examining this redundant design carefully, it becomes obvious that sensing devices are necessary to detect each failure, and switching mechanisms are reqı ired to activate and deactivate elements. In practical situations, the present limits on the reliability of conventional sensing and switching devices limit the reliability potential of the scheme.

2.5 Standby Redundancy: D operating Units Necessary for Survival.

Another form of binomial redunancy may be encountered when the system (subsystem) is composed of N identical units. A subset of D units out of N is chosen to form a working system (subsystem). Elements of the working system (subsystem) which fail may be replaced. The units which are not a part of the working system are assumed to be standby units and have a failure rate $\lambda = 0$. It is assumed that failure-free switching, sensing, and interconnection methods are utilized in order to determine an upper bound reliability for this system type.

In this type of redundant system, since D units are always working, the failures of this subsystem form a Poisson process with rate λD. Then the system reliability will be the probability that the system has a maximum number (N-D) of failures in the time period t, i.e.:

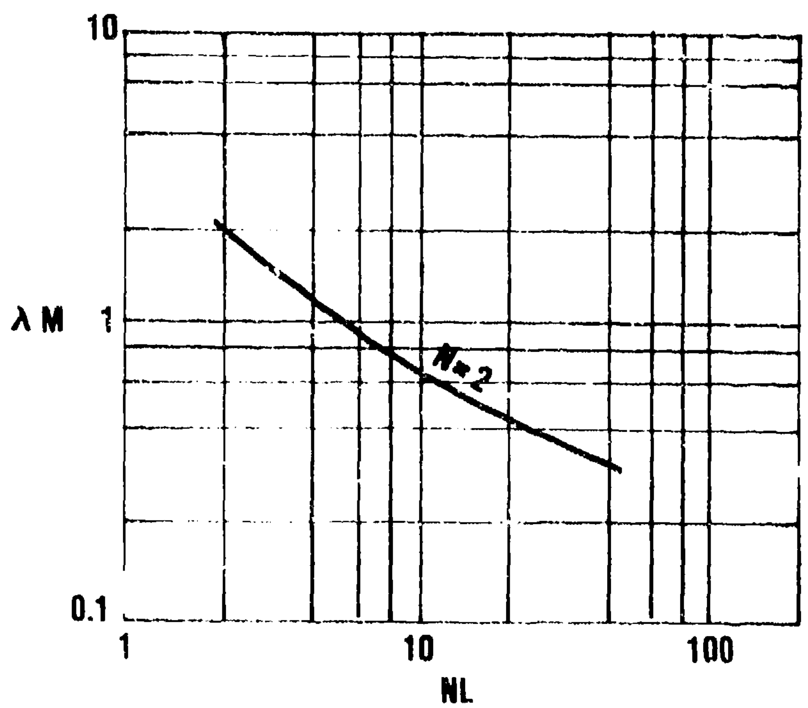
$$R(t) = \sum_{k=0}^{N-D} P(N(t) = k)$$

$$= \sum_{k=0}^{N-D} \frac{e^{-D\lambda t}(D\lambda t)^k}{k!}$$

$$= e^{-D\lambda t} \sum_{k=0}^{N-D} \frac{(D\lambda t)^k}{k!} \qquad (2.18)$$

where

N(t)= the number of failures in the time period t.
N= total number of elements composing the system.
D= number of elements necessary for the system to operate.
λ= failure rate of each operating element.



NL = total number of units in system

A figure of improvement of redundant system mean life over simplex system mean life may be found by multiplying the value determined for λM by L. λ= failure rate of a single system unit: M = system mean life.

Figure 2.5. λM Relationship for Standby Redundancy

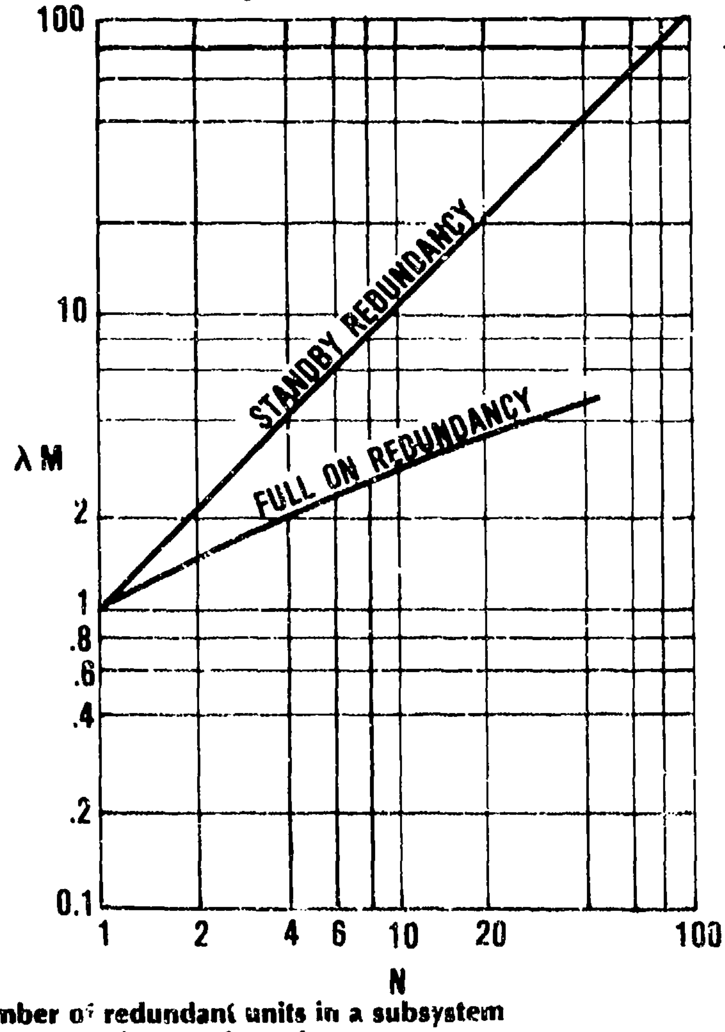
2.6 Reliability of Parallel Units when the Reliability of Switching is Considered.

In the previous sections, we have discussed the reliability of non-repairable redundant systems, ignoring the failure rate of any sensing or switching mechanisms. As such, the previous evaluation and analysis procedures should be considered as providing an upper bound for system reliability.

Various ways have been suggested through which the failure rates of sensing and switching devices can be accounted for in the analysis of redundant systems. These range from simply adding a failure rate increment equal to the failure rate of a switching and sensing device to the failure rate of one or more redundant units, to analysis procedures which take into account the operational modes of the sensing and switching device. The following is an example of the latter (for a two-unit redundant system) which provides information on how such an analysis may be performed, and also provides some insight into the complexities of the analysis.

Consider units A and B connected in a standby parallel configuration. If either A or B is functioning and properly connected, the required system function is realized. The sensor/switch S provides the necessary connection, disconnection function.

If A fails, S senses this failure, and if S is operating properly it switches to B. The system composed of A,B, and S operates as follows:



N = number of redundant units in a subsystem
λ = failure rate of any active subsystem
M = mean life of a subsystem

Figure 2.6 Comparison of Standby and Full On Redundancy

a. If S operates properly, it checks A. If A has failed, it turns on B. S is then in a fail safe position. The system operates until B fails.
b. S fails (and no switch possible) while A is operating. The system operates until A fails.
c. S fails in a way that a switch to B is mandated, while A is still capable of operating. B is energized and the system operates until B fails.
d. S fails while A is still operating. It fails in such a way that A and B are unable to operate and the system fails.

Let $P_a(t)$, $P_b(t)$, $P_c(t)$, and $P_d(t)$ denote the probability that the system fails at time t according to the above events, a,b,c, and d, respectively. Then, noting that the above events are all mutually exclusive at t, we have the probability of failure at t:

$$P_F(t) = P_a(t) + P_b(t) + P_c(t) + P_d(t)$$

Let the density functions of time to failure for A,B,S, be exponential and let A and B have identical density functions.

$$f(t) = \lambda e^{-\lambda t} \qquad \text{for A and B}$$

and

$$g(t) = \lambda_0 e^{-\lambda_0 t} \qquad \text{for S}$$

Note that (b), (c) and (d) indicate three different modes of possible failure predicated on the single failure density function. In order to cope with this, define:

P_1 = probability when S fails, the switch stays on A
P_2 = probability when S fails, the switch goes to B
P_3 = probability when S fails, the switch makes A & B inoperative.

The events described by P_1, P_2, P_3 are also mutually exclusive and exhaustive and:

$$P_1 + P_2 + P_3 = 1$$

We will now develop probabilistic relationships for $P_a(t)$, $P_b(t)$, $P\ (t)$ and $P_d(t)$:
Let:

t_s = the time at which S fails.
t_a = the time at which A fails.
t_b = the time at which B fails.

Noting that:

$$P(t_g) = 1 - \int_0^{t_a} g(t_g)\, dt_g$$

= probability that S does not fail before t_s,

$$P(t_s) = 1 - \int_0^{t_s} f_s(t_a)\, dt_a$$

= probability that A doesn't fail before t_a, we have:

$$P_a(t) = \int_{t_b=0}^{t} \int_{t_a=0}^{t-t_b} P(t_a)\, f_a(t_a)\, f_b(t_b)\, dt_a\, dt_b .$$

This follows from:

$$W(t-t_b) = \int_{t_a}^{t-t_b} P(t_a)\, f_a(t_a)\, dt_a$$

= probability that A fails before S fails in $[0, t-t_b]$,

and:

$\int_{t_b}^{t} W(t-t_b) f_b(t_b)\, dt_b =$ probability that B fails in $[0,t]$ after the switch was made upon the failure of A.

By the assumption of exponential failure, we have:

$$P_a(t) = \frac{\lambda}{\lambda+\lambda_0}\left[1 - e^{-\lambda t} - \frac{\lambda}{\lambda_0}\left(e^{-\lambda t} - e^{-(\lambda+\lambda_0)t}\right)\right]. \tag{2.19}$$

Using the memoryless property of the exponential distribution, it can be seen that A has the same density as before, after S fails.

Then we have:

$$P_b(t) = P_1 \int_{x=0}^{t} \int_{t_s=0}^{t-x} P(t_s)\, g(t_s)\, f_a(x)\, dt_s\, dx.$$

This follows from the following:

$$W(t-x) = \int_{t_s=0}^{t-x} P(t_s)\, g(t_s)\, dt_s$$

= probability that S fails before A fails in $[0, t-x]$

and it follows:

$$P_b(t) = P_1 \int_{x=0}^{t} W(t-x)\, f_a(x)\, dx$$

= probability that A fails in $[0,t]$ after S fails.

Similarly we have:

$$P_c(t) = P_2 \int_{t_b=0}^{t} \int_{t_s=0}^{t-t_b} P(t_s)\, g(t_s)\, f_b(t_b)\, dt_s\, dt_b$$

and

$$P_d(t) = P_3 \int_{t_s=0}^{t} P(t_s)\, g(t_s)\, dt_s .$$

The assumption of the exponential density gives:

$$P_b(t) = P_1 \left[\frac{\lambda_o}{\lambda+\lambda_o} - e^{-\lambda t} + \frac{\lambda}{\lambda+\lambda_o} e^{-(\lambda+\lambda_o)t} \right] \qquad (2.20)$$

$$P_c(t) = P_2 \frac{\lambda_o}{\lambda+\lambda_o} \left[1 - e^{-\lambda t} - \frac{\lambda}{\lambda_o} \left[e^{-\lambda t} - e^{-(\lambda+\lambda_o)t} \right] \right] \qquad (2.21)$$

$$P_d(t) = P_3 \frac{\lambda_o}{\lambda+\lambda_o} \left[1 - e^{-(\lambda_o+\lambda)t} \right]$$

From the above results the system reliability can be obtained as follows:

$$R(t) = 1 - P_F(t) = 1 - P_a(t) - P_b(t) - P_c(t) - P_d(t)$$

$$= 1 - \frac{\lambda}{\lambda + \lambda_o}\left[1 - e^{-\lambda t} - \frac{\lambda}{\lambda_o}[e^{-\lambda t} - e^{-(\lambda+\lambda_o)t}]\right]$$

$$- (P_1 + P_2)\left[\frac{\lambda_o}{\lambda+\lambda_o} - e^{-\lambda t} + \frac{\lambda}{\lambda+\lambda_o} e^{-(\lambda+\lambda_o)t}\right]$$

$$- P_3 \frac{\lambda_o}{\lambda+\lambda_o}\left[1 - e^{-(\lambda+\lambda_o)t}\right] \qquad (2.22)$$

If $P_3 = 0$, $P_1 + P_2 = 1$ (fail safe provisions built into system such that a switch/sense failure cannot cause system to directly fail.)

$$R(t) = e^{-\lambda t} + \frac{\lambda}{\lambda_o} e^{-\lambda t} (1 - e^{-\lambda_o t}) \qquad (2.23)$$

$$M = \int_0^\infty R(t)\, dt = \frac{1}{\lambda} + \frac{1}{\lambda_o} - \frac{\lambda}{\lambda_o(\lambda_o+\lambda)} \qquad (2.24)$$

Note that by (2.23) and (2.24)

$$\lim_{\lambda_o \to o} R(t) = \lim_{\lambda_o \to o} e^{-\lambda t} + \frac{\lambda}{\lambda_o} e^{-\lambda t} (1-e^{-\lambda_o t})$$

$$= e^{-\lambda t} + \lim_{\lambda_0 \to 0} [\lambda t e^{-(\lambda + \lambda_0)t}] \qquad (2.25)$$

$$= (1 + \lambda t) e^{-\lambda t}$$

$$\lim_{\lambda_o \to o} M = \lim_{\lambda_o \to o} \frac{1}{\lambda} + \frac{1}{\lambda_o} - \frac{\lambda}{\lambda_o (\lambda_o+\lambda)}$$

$$M = \frac{1}{\lambda} + \lim_{\lambda_0 \to 0} \frac{\lambda_0}{\lambda_0 (\lambda_0 + \lambda)}$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda} \qquad (2.26)$$

$$= \frac{2}{\lambda} \; ,$$

which are identical to (2.12) and (2.13), respectively, for a two-unit standby redundant system.

CHAPTER 3

RELIABILITY OF REPAIRABLE SYSTEMS

The previous section was concerned with the reliability of redundant systems which were not maintained; that is, the assumption was made that the redundant configuration was maximally operative at time zero and no unit repairs were performed until the system failed. At that time, all the units were repaired or replaced and the system put in a maximum operative condition again. This section makes the assumption that as units of a redundant system fail, they may be repaired. In particular, we will assume that the time to failure of each unit and the time to repair of each unit follow exponential density functions.

$$r(t) = \lambda e^{-\lambda t} \quad \text{for time to failure}$$
$$g(t) = \mu e^{-\mu t} \quad \text{for time to repair}$$
$$\lambda = \text{failure rate}$$
$$\mu = \text{repair rate}$$

In the previous section we were concerned only with two reliability figures of merit: (1) Reliability expressed as a probability P(t) which denotes either the probability of operating satisfactorily over a period of time (0,T) or the probability that the system will be operating satisfactorily at the end of a period of time (0,T), T. (2) Reliability expressed as mean time to first system failure M. Since the concept of maintained systems forces a change in the operational scenario, the figures of merit of interest are somewhat modified and augmented. The figures of merit with which we will be concerned will be as follows:

(1) Reliability expressed as the probability that the system will be operative at any time t, P(t).
(2) System mean time to first failure M (defined in the same way as in the non-repairable case), and system steady state mean time to failure M_s.
(3) The expected fractional amount of time that the system will be functional during a period of time (0,T) - (Note that this figure of merit could be applied to the non-repairable case, however, it is rarely used and it is not as meaningful as it is in the repairable case).
(4) Reliability expressed as the probability that the system will not fail during the time period (0,T).

It is interesting to note that while (1) and (3) are mathematically different, both are commonly referred to as "Availability" (actually while the development of both measures is different and their "time oriented" forms are different, their limiting cases are identical.)

In general, the procedure which must be followed in the analysis and evaluation of a repairable redundant system is the definition of its relevant system states followed by an analysis of possible transitions. This means that the system can be in any one of various states $E_0, E_1, E_2 \ldots E_N$. For example, a single unit has two possible states (1) operating and (2) failed. A two-unit redundant system as in Figure 3.1 has three possible states: (1) both units on, (2) one unit on, one unit failed, (3) both units failed. Further, the system can pass from adjacent state to adjacent state at rates defined by the state's failure and repair rates. (For purposes of clarity, let us define adjacent state as that state accessible to another state by a single repair or a single failure. For example, in the case of the two redundant units in Figure 3.1, the state of both units on is not adjacent to the state of both units failed - the state of both units on is adjacent to the state of one unit on, one unit failed). It is impossible to pass from one state to another unless a chain path of adjacent states is established. Given such rules, many means are available to analyze redundant repairable systems; some are more complex than others. This section will discuss several of these means.

3.1 Analysis of a Single Unit.

To start, we will discuss the reliability of a single unit. As will be seen later, the reliability expressions for a single unit can serve as time saving building blocks for the definition of the reliability expressions for complex redundant repairable systems.

Consider a unit having a failure rate λ and a repair rate μ.

Let $P_{ij}(t)$ denote the probability of a unit being in state j at time t given it was in state i at t= 0.

0 = an operating state (an up state)

1 = a failed state (a down state)

$i, j = \}^{0}_{1}$

then:

$P_{00}(t+\Delta t) = P$ (unit is operating at $t+\Delta t$, given it was up at $t = 0$)

$=$ P(unit doesn't fail in $(t,t+\Delta t)$ | unit is up at t, given it was up at $t = 0$) $\cdot$ P (unit is up at t, given it was up at $t = 0$)

$+$ P(a repair is completed in $(t,t+\Delta t)$ | unit is down at t, given it was up at $t = 0$) $\cdot$ P(unit is down at t, given up $t = 0$)

$= (1-\lambda\Delta t)\, P_{00}(t) + \mu\Delta t\, P_{01}(t)$

Using:

$$P_{00}(t) + P_{01}(t) = 1,$$

$$P_{00}(t+\Delta t) = (1-\lambda\Delta t)P_{00}(t) + \mu\Delta t(1 - P_{00}(t))$$

$$\frac{P_{00}(t+\Delta t) - P_{00}(t)}{\Delta t} = \mu - (\lambda+\mu)\, P_{00}(t) = \frac{dP_{00}(t)}{dt}$$

Solving this differential equation yields:

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + Ke^{-(\lambda+\mu)t}$$

Using $P_{00}(0)=1$, then $K=\lambda/(\lambda+\mu)$.
Then:

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t} \qquad (3.1)$$

In a similar fashion the expression for $P_{10(t)}$ can be derived using $P_{10}(0)=1$, (i.e., unit is down at $t=0$) then

$$K = -\frac{\mu}{\lambda+\mu}.$$

Thus:

$$P_{10}(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \tag{3.2}$$

Since $P_{01}(t)=1-P_{00}(t)$ and $P_{11}(t)=1-P_{10}(t)$

$$P_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \tag{3.3}$$

$$P_{11}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \tag{3.4}$$

The relationships (3.1) to (3.4) are composed of a constant term ($\frac{\mu}{\lambda+\mu}$) or $\frac{\lambda}{\lambda+\mu}$ and a term which varies with time ($Ke^{-(\lambda+\mu)t}$). As the time period in time gets arbitrarily large, we get:

$$e^{-(\lambda+\mu)t} \to 0$$

and:

$$P_{00}(t) \to \frac{\mu}{\lambda+\mu} \left[\text{and } P_{01}(t) \to \frac{\lambda}{\lambda+\mu}\right] \tag{3.5}$$

$$P_{10}(t) \to \frac{\mu}{\lambda+\mu} \left[\text{and } P_{11}(t) \to \frac{\lambda}{\lambda+\mu}\right] \tag{3.6}$$

These denote the limiting probability that the unit will be operating at any arbitrary point in time distant from $t=0$. It will be noted that the relationships above indicate that as the time of interest t becomes distant from $t=0$, the original state of the unit is of no consequence. The stochastic behavior of the transition probability $P_{ij}(t)$ is shown in Figure 3.2. Equations (3.5) and (3.6) above are defined as Availability given by:

$$A=\frac{\mu}{\lambda+\mu} = \frac{M}{M+\bar{R}} \tag{3.7}$$

where: $\frac{1}{\mu} = \bar{R}$ = meantime to repair of the unit

$\frac{1}{\lambda} = M$ = meantime to failure of the unit.

Equations (3.1) to (3.4) define the time dependent Availability of a unit, P(t). The concept of expectation indicates that:

$$\int_0^T P(t)\, dt \tag{3.8}$$

defines the mean up or operational time, over a period of time (0,T) realizing a unit can fail, be repaired, fail again, etc. It then follows that the expected fraction of time that the unit will be operating during the interval (0,T) is given by:

$$E(F)= \frac{1}{T} \int_0^T P(t)\, dt \tag{3.9}$$

From equation (3.1):

$$E_{00}(F)= \frac{1}{T} \int_0^T P_{oo}(t)dt= \frac{1}{T} \int_0^T \left(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}\right)dt$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{T(\lambda+\mu)^2} - \frac{\lambda}{T(\lambda+\mu)^2} e^{-(\lambda+\mu)T} \tag{3.10}$$

which denotes the expected fractional amount of time that the unit is on in the interval (0,T) given that the unit is operating at t=0. Similarly from equation (3.2):

$$E_{10}(F) = \frac{1}{T} \int_0^T F_{10}(t)\,dt$$

$$= \frac{\mu}{\lambda+\mu} - \frac{\mu}{T(\lambda+\mu)^2} + \frac{\mu}{T(\lambda+\mu)^2} e^{-(\lambda+\mu)T} \qquad (3.11)$$

which denotes the expected fractional amount of time that the unit is on in the interval (0,T) given that the unit is down at t=0.

Both relationships (3.10) and (3.11) are composed of a constant term $\frac{\mu}{\lambda+\mu}$ and terms which vary with the time interval in question. As the time period (0,T) gets arbitrarily large, these equations reduce to the limiting case such that the expected fractional amount of time that the unit is on during a very large period of time is:

$$E_{\infty}(F) = E_{10}(F) = \frac{\mu}{\lambda+\mu} \qquad (3.12)$$

Again, it will be noted that the relationships above indicate that as the time of interest t becomes distant from t=0, the original state of the unit is of no consequence. Note that relationship (3.12) is identical to relationship (3.7). Hence Availability is defined as either:

(a) The expected fractional amount of time that the unit is operating over an arbitrarily long period of time.

(b) The probability that the unit is operating at any point in time distant from t=0.

In either case, then

$$A = \frac{\mu}{\mu+\lambda} = \frac{M}{M+\bar{R}} .$$

The fact that availability can have a dual definition is obvious when definition (a) is considered first, for if a unit can be expected to be operational P percent of the time and is capable of numerous repairs and failures, then it follows that at any random point in time the probability is P percent that the unit is operating.

Relationship (3.7) was developed using a differential equation and assumptions relating to the distributions of repair and failure times. It need not be developed by that means, using such assumptions. It may be developed quite simply and non-parametrically as follows:

Let:

T = the interval of time in question
T_0 = the time during the interval that the unit is operating
T_1 = the time during the interval that the unit is down

Then:

$$A = \frac{T_0}{T} \quad \text{(definition (a) of availability)}$$

But:

$$T = T_0 + T_1$$

$$\frac{T_0}{M} = \text{number of failures expected in operating time } T_0$$

$$T_1 = \frac{T_0}{M}\bar{R} = \text{time that the unit is down over interval in question}$$

$$\therefore A = \frac{T_0}{T_0 + \frac{T_0 \bar{R}}{M}} = \frac{1}{1 + \frac{\bar{R}}{M}} = \frac{M}{M+\bar{R}} \qquad (3113)$$

which is identical to (3.7)
Hence the limiting relationship for Availability is non-parametric.

There will be situations where evaluation requires the use of one or more of the measures previously discussed. In summary, the most important are repeated and defined below:

(a) $$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

The probability a unit will be operating satisfactorily at a given point in time, given that the unit was operating satisfactorily at t=0.

(b) $$E_{00}(F) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{T(\lambda+\mu)^2} - \frac{\lambda}{T(\lambda+\mu)^2} e^{-(\lambda+\mu)T}$$

The expected fractional amount of time that the unit is on in the interval (0,T), given that the unit is operating at t= 0.

(c) $$A = \frac{\mu}{\mu+\lambda} = \frac{M}{M+\bar{R}}$$

The probability a unit will be operating at a random point in time, distant from t=0, or the expecte fractional amount of time that the unit is on during an arbitrarily long time interval (0,T).

3.2 The Combined Units Approach: (Full on Redundancy)

This can be considered one of the most basic approaches in existence. It starts with the definition of $P_{00}(t)$ or Availability, for a single unit and treats these as basic probabilities of survival, as discussed in Chapter 1 (in a reliability block diagram).

Take, for example, a two-unit parallel system as in Figure 1.1. Both units are identical and originally operable (on at t= 0) and both are operating simultaneously. If one of the units fails, repairs are begun on it and the other unit performs the function. As soon as the failed unit is repaired, it is returned to operation. At the first instant of time when both of the units are failed, the system has failed.

The probability that Unit A is operating at t= $P_{00}(t)$

The probability that Unit B is operating at t= $P_{00}(t)$

The probability that either A or B or both are operating at t is the Availability of the system P(t).

$$P(t) = P_{00}(t) + P_{00}(t) - (P_{00}(t))^2$$

$$= 2\,P_{00}(t) - P_{00}(t)^2$$

$$= 2\left[\frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\;e^{-(\mu+\lambda)t}\right] - \left[\frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\;e^{-(\mu+\lambda)t}\right]^2$$

which when expanded and combined reduces to:

$$P(t) = \frac{\mu^2 + 2\lambda\mu}{(\lambda+\mu)^2} - \frac{\lambda^2 e^{-2(\lambda+\mu)t}}{(\lambda+\mu)^2} + \frac{2\lambda^2\;e^{-(\lambda+\mu)t}}{(\lambda+\mu)^2} \qquad (3.14)$$

Note that as t gets arbitrarily large (3.14) reduces to:

$$P(t) = \frac{\mu^2 + 2\lambda\mu}{(\lambda+\mu)^2} = A_{system} \qquad (3.15)$$

which is the limiting availability for the two-unit redundant case. To show this directly define as before:
The probability that unit A and unit B will be operating at a random point in time, distant from $t=0$, as respectively:

$$A_A = \frac{\mu}{\mu+\lambda} \text{ and } A_B = \frac{\mu}{\mu+\lambda}$$

The probability that either A or B or both are operating at the same random point in time is:

$$A_{system} = A_A + A_B - A_A A_B = \frac{\mu}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} - \left(\frac{\mu}{\mu+\lambda}\right)^2$$

which reduces to:

$$A_{system} = \frac{\mu^2 + 2\lambda\mu}{(\lambda+\mu)^2} \tag{3.16}$$

This is identical to (3.15).

In general, this approach is one of the simplest to use when an evaluation of either the limiting or the time dependent availability of a system or subsystem is required (mean time to first failure or Probability of no system failure over time t cannot be evaluated using this approach).

The following formula may be utilized to determine the system or subsystem redundancy and general philosophy of operation as discussed in the beginning of this section.
Let X_i= time dependent or limiting availability of any unit.

$$\text{System Availability} = \prod_{i=1}^{L} \sum_{j=D_i}^{N_i} \binom{N_i}{j} X_i^j (1-X_i)^{N_i-j} \tag{3.17}$$

where:

- L= number of subsystems in series
- i= defines the i^{th} subsystem
- D= minimum number of working units required for the i^{th} subsystem to operate
- N_i= total number of parallel units comprising the i^{th} subsystem.

3.3 The Markovian Approach (For Full on Redundancy)

Of all the approaches to the analysis of repairable redundant configurations, the Markovian is the most powerful. It is particularly appropriate to the analysis of redundant systems and through its application to such characteristics as:

. Availability (time dependent and limiting)
. Mean Time between system failure
. Reliability (we saw earlier a simple Markovian approach analysis for a non-repairable system).

A redundant repairable system composed of N units has 2^N potential states. Each time a different combination of units is failed and operating a new state is defined. This means for every single repair action completed, the system enters a new state. Likewise, for each failure which occurs, the system enters a new state. The repairs and failures which occur cause the transitions from one state to the next. The rates of transition (the failure rates, and repair rates) from one state to the next are determined by the failure rate and repair rate characteristics of the current state and also the probability of more than one transition occurring simultaneously is zero.

In order to apply the procedure, all possible system states must be identified and probabilistic equations developed describing such states. The easiest way to define and write the probabilistic relations of the states is to draw a state space diagram or a truth table. The diagram shows visually the evolution of different system states possible and the means of transition, if any, between states, either by failure or repair. Because small increments of time are considered in the analysis, the probability of a double transition is considered to be zero. The truth table (Figure 3.2) shows the results of the space diagram in tabular form. Examples of both will follow.

The definition of:

(A) Availability (time dependent, and the limiting case)
(B) Reliability and Mean time to system failure

require the utilization of slightly different constraints and formulation of slightly different sets of state equations. The primary difference lies in the fact that if we wish to determine the probability that at any time the system is in any state K (necessary to determine Availability) we must allow for a transition from a system failed state to an operating state. In the event we wish to determine the reliability (the probability of no system failure in an interval (0,T)) that is, the probability that the system remains in the set of non failed states during the interval (0,T),

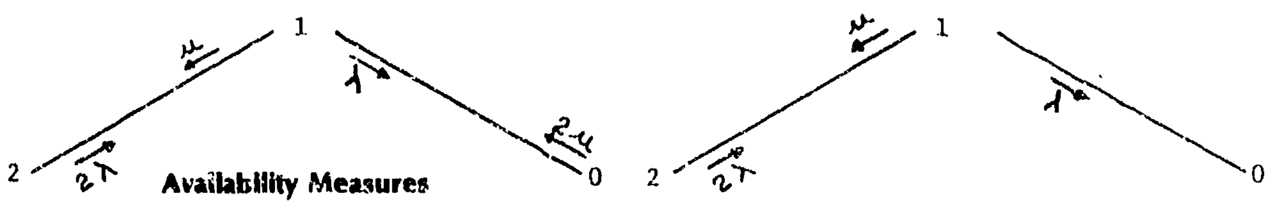
$$R(t) = \sum_{\Omega} R_i(t) \qquad \Omega = \text{Set of all non failed states.}$$

or the mean time to failure of the system, we must structure our state equations such that there is no transition from a failed state.

The space diagram for a two-unit system (both units operating simultaneously, only one of which is required for systems operation) is shown in figure 3.1. As can be seen, the possible states are:

a) Unit A and Unit B are both operating (State 2)
b) One unit is in a failed state (repairs are being made) and the other is successfully operating (State 1)
c) Both units are failed (State 0)

The truth table can be tabulated as in Figure 3.2.
Values of the λ's and μ's define transition probabilities between adjacent states. The arrows indicate which direction the transitions take.



State Space Diagram For Reliability And Mean Time To Failure Measures
Figure 3.1

STATE	A	B	SYSTEM
2	1	1	Operating
1	0	1	Operating
1	1	0	Operating
0	0	0	Failed

Truth Table For A Redundant System

Figure 3.2

3.3.1 Markovian Approach for Availability Measures (Time Dependent and Limiting Case) (Full on Redundancy)

The following set of state equations may be developed

$$
\begin{aligned}
P_2(t+\Delta t) &= P\ (\text{unit A and B are both operating at } t+\Delta t) \\
&= P\ (\text{neither A nor B fail in } (t,t+\Delta t) \mid \text{both units are operating at } t) \cdot P(\text{both units are operating at } t) \\
&+ P\ (\text{the repair of A(B) is completed in } (t,t+\Delta t) \mid \text{B(A) is operating at } t) \cdot P(\text{B(A) is operating at } t) \\
&= (1-2\lambda\Delta t)\ P_2(t) + \mu\Delta t\ P_1(t)
\end{aligned}
\tag{3.18}
$$

Similarly, we have:

$$P_1(t+\Delta t) = 2\lambda\Delta t\ \ P_2(t) + \left[1-(\lambda+\mu)\Delta t\right] P_1(t) + 2\mu\Delta t\ \ P_0(t) \tag{3.19}$$

$$P_0(t+\Delta t) = \lambda\Delta t\ P_1(t) + \left[1-2\mu\Delta t\right] P_0(t) \tag{3.20}$$

where:

$$P_0(t) + P_1(t) + P_2(t) = 1 \tag{3.21}$$

Expanding and rearranging the state equations as follows:

$$\frac{P_2(t+\Delta t) - P_2(t)}{\Delta t} = \frac{dP_2(t)}{dt} = \mu P_1(t) - 2\lambda P_2(t) \tag{3.22}$$

$$\frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = \frac{dP_1(t)}{dt} = -(\lambda+\mu)P_1(t)$$

$$+ 2\lambda P_2(t) + 2\mu P_0(t) \tag{3.23}$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = \frac{dP_0(t)}{dt} = \lambda P_1(t) - 2\mu P_0(t)$$

$$\tag{3.24}$$

Taking Laplace transforms and realizing that:

$$P_2(0) = 1, \quad P_1(0) = 0, \quad P_0(0) = 0$$

(initial conditions if we assume all units operative at $t = 0$).

$$s\ P_2(s) - 1 = \mu P_1(s) - 2\lambda P_2(s)$$

$$s\ P_1(s) = -(\lambda+\mu)\ P_1(s) + 2\lambda\ P_2(s) + 2\mu P_0(s)$$

$$s\ P_0(s) = \lambda P_1(s) - 2\mu\ P_0(s)$$

Solution of these simultaneous equations and finding the inverse transformation (as shown in Appendix B) results in:

$$P(t)= 1- P_0(t)= \frac{\mu^2+2\lambda\mu}{(\lambda+\mu)^2} - \frac{\lambda^2 e^{-2(\lambda+\mu)t}}{(\lambda+\mu)^2} + \frac{2\lambda^2 e^{-(\lambda+\mu)t}}{(\lambda+\mu)^2} \tag{3.25}$$

and taking the limiting case (letting $t \to \infty$)

$$P(t) \to A= \frac{\mu^2+2\lambda\mu}{(\lambda+\mu)^2} \tag{3.26}$$

Note that results (3.25) and (3.26) are identical to results (3.14) and (3.15) and (3.16) which came about as a consequence of an entirely different approach to the problem (an approach which was considerably less involved than the classical Markovian approach discussed previously).

3.3.2 Markovian Approach for Reliability Measures and Mean Time to System Failure (Full on operation)

The following set of state equations may be developed (derivation similar to previous, except notice that once a unit reaches a failed state $P_0(t)$ it is not allowed to pass to a working state $P_1(t)$).

$$P_2(t+\Delta t)= P_2(t)\ (1-2\lambda\Delta t) + P_1(t)\mu\Delta t \tag{3.27}$$

$$P_1(t+\Delta t)= P_2(t)\ 2\lambda\Delta t + P_1(t)\ (1-(\lambda+\mu)\Delta t) \tag{3.28}$$

$$P_0(t+\Delta t)= P_1(t)\lambda\Delta t + P_0(t) \tag{3.29}$$

$$P_2(t) + P_1(t) + P_0(t)= 1 \tag{3.30}$$

As before manipulation of the state equations results in the differential equations:

$$\frac{dP_2(t)}{dt} = -2\lambda P_2(t) + \mu P_1(t) \tag{3.31}$$

$$\frac{dP_1(t)}{dt} = 2\lambda P_2(t) - (\lambda+\mu)P_1(t) \tag{3.32}$$

$$\frac{dP_0(t)}{dt} = \lambda P_1(t) \tag{3.33}$$

Taking Laplace transforms and realizing that:

$$P_2(0) = 1,\ P_1(0) = 0,\ P_0(0) = 0$$

(initial conditions if we assume all units operational at t= 0) we obtain:

$$1 = (s+2\lambda)\ P_2(s) - \mu P_1(s) \tag{3.34}$$

$$0 = 2\lambda P_2(s) - (\lambda+\mu+s)\ P_1(s) \tag{3.35}$$

$$0 = sP_0(s) - \lambda P_1(s) \tag{3.36}$$

(in a manner similar to Appendix B). The same argument of the previous section results in:

$$R(t) = \frac{s_1 e^{s_2 t} - s_2 e^{s_1 t}}{s_1 - s_2}$$

$$s_1,\ s_2 = -\frac{(3\lambda+\mu) \pm (\lambda^2+6\mu\lambda+\mu^2)^{1/2}}{2} \tag{3.37}$$

as before:

$$M = \int_0^\infty R(t)\ dt$$

$$= \frac{3\lambda+\mu}{2\lambda^2} \tag{3.38}$$

(assuming the system at $t=0$ had both units in satisfactory operation). Note that if $\mu=0$ (that means non-repairable system)

$$M = \frac{3}{2}\frac{1}{\lambda} \tag{3.39}$$

which is exactly the value which would have resulted from relationship (2.6) for 2 units in a non-repairable system, and if $\mu=0$ relationship (3.37) reduces to:

$$R(t) = 2e^{-\lambda t} - e^{-2\lambda t} \tag{3.40}$$

which is exactly the value which would have resulted from relationship (2.7) for 2 units in a non-repairable system.

3.4 The Markovian Approach For Stand-by Redundancy, Markovian Approach for Availability Measures (Time Dependent and Limiting Case) (System in Stand-by Conditions)

In this situation our intent is to setup the probability state equations relevant to the case where the redundant configuration has one unit in actual operation and the other units are in stand-by (such units are not energized) and have failure rates = 0. The unit in operation operates until it fails, at which time one of the stand-by units begins operation, and repairs are begun on the failed unit. When the failed unit has been repaired, it becomes a stand-by unit. When a failure occurs when no repaired (or good) stand-by units are available, the system fails.

As before, an example is presented as to how such a problem is solved. We take again a two-unit redundant system and define its respective states.

(1) Unit A and Unit B, both are operable - one is operating (State 2)
(2) One unit is in a failed state (repairs are being made) and the other is operable (State 1)
(3) Both A and B are in a failed state (State 0). Then we have the following state equations:

$$P_2(t+\Delta) = P_2(t)(1-\lambda\Delta t) + P_1(t)\mu\Delta t \tag{3.41}$$

$$P_1(t+\Delta t) = P_2(t)\lambda\Delta t + P_1(t)(1-(\lambda+\mu)\Delta t) + P_0(t)2\mu\Delta t \tag{3.42}$$

$$P_0(t+\Delta t) = P_1(t)\,\lambda\Delta t + P_0(t)\,(1-2\mu\Delta t) \tag{3.43}$$

Expanding and rearranging the equations:

$$\frac{dP_2(t)}{dt} = \mu P_1(t) - \lambda P_2(t)$$

$$\frac{dP_1(t)}{dt} = -(\lambda+\mu)\ P_1(t) + \lambda P_2(t) + 2\mu P_0(t)$$

$$\frac{dP_0(t)}{dt} = \lambda P_1(t) - 2\mu P_0(t)$$

Taking Laplace transforms and solving for:

$$P(t) = 1 - P_0(t)$$

$$P(t) = 1 - \frac{\lambda^2}{s_1 s_2} + \frac{\lambda^2 e^{s_2 t}}{s_2(s_2 - s_1)} + \frac{\lambda^2 e^{s_1 t}}{s_1(s_1 - s_2)} \qquad (3.44)$$

$$s_1, s_2 = \frac{-(2\lambda + 3\mu) \pm \sqrt{\mu^2 + 4\mu\lambda}}{2}$$

Note as t gets arbitrarily large:

$$P(t) = A = \frac{2\mu^2 + 2\mu\lambda}{2\mu^2 + \lambda^2 + 2\mu\lambda} \qquad (3.45)$$

3.4.1 Markovian Approach for Reliability and Mean Time to Failure Measures (System in Stand-by operation)

Assume a redundant stand-by configuration and operating philosophy as described in the previous section. Again, the primary objective is to develop the probability state equations representing the system, its operational philosophy and the measures to be evaluated.

As before an example is presented as to how such a problem is solved. We take again a two-unit system and define its possible states:

(1) One unit is operating and has a failure rate= λ, a second unit is capable of operation but is not energized and has a failure rate= 0. (State 2)

(2) One unit is in a failed state (repairs are being made) and the other is successfully operating (State 1)

(3) Both units are failed (incapable of operation due to malfunction) (State 0).

Note as previously stated, that in order to evaluate Reliability or Mean Time to Failure, the state equations can permit no transition from a failed state. As before, state equations are developed.

$$P_2(t+\Delta t)= P_2(t)(1-\lambda\Delta t) + P_1(t)\,\mu\Delta t \qquad (3.46)$$

$$P_1(t+\Delta t)= P_2(t)\lambda\Delta t + P_1(t)\left[1-(\lambda+\mu)\Delta t\right] \qquad (3.47)$$

$$P_0(t+\Delta t)= P_1(t)\lambda\Delta t + P_0(t) \qquad (3.48)$$

After solving the equations finding the inverse transformation and performing manipulation

$$R(t)= \frac{r_1 e^{r_2 t} - r_2 e^{r_1 t}}{r_1 - r_2} \qquad (3.49)$$

where:

$$r_1, r_2 = \frac{-(\mu+2\lambda) \mp \sqrt{(\mu+2\lambda)^2 - 4\lambda^2}}{2}$$

Again, since $r_1, r_2 < 0$ and $r_1 > r_2$, we have:

$$M = \int_0^{\infty} R(t)\,dt = \int_0^{\infty} \frac{r_1 e^{r_2 t} - r_2 e^{r_1 t}}{r_1 - r_2}\,dt$$

$$= \frac{r_1}{(r_1 - r_2)} \int_0^{\infty} e^{r_2 t}\,dt - \frac{r_2}{(r_1 - r_2)} \int_0^{\infty} e^{r_1 t}\,dt$$

$$= -\frac{r_1}{r_2(r_1 - r_2)} + \frac{r_2}{r_1(r_1 - r_2)}$$

$$= -\frac{r_1 + r_2}{r_1 r_2}$$

$$= \frac{2\lambda + \mu}{\lambda^2} \qquad (3.50)$$

Note if the system is non-repairable, $\mu = 0$, then

$$M = \frac{2}{\lambda} \qquad (3.51)$$

which is identical to the result of Equation (2.14) for a two unit non-repairable system.

By this time it is clear that the Markovian approach is capable of evaluating all of the reliability measures required. However, it is also clear that this approach is rather cumbersome and time-consuming, especially when more than two units are redundant or when the units have different values of λ and μ. Note also that the Markovian approaches discussed are designed to evaluate specific configurations/operational scenarios, and not to provide closed form relationships, relating failure and repair rates of units, number of units in a subsystem and number of subsystems to system reliability. The next two sections provide discussions of two approaches which are capable of making the analysis of repairable subsystems less cumbersome and time-consuming than those discussed and have the additional characteristic of being in closed form or algorithmic form such that a single equation, or procedure is capable of evaluating any number of units in parallel.

3.5 State Expectation/Transition Model (for Reliability and Mean time to failure)

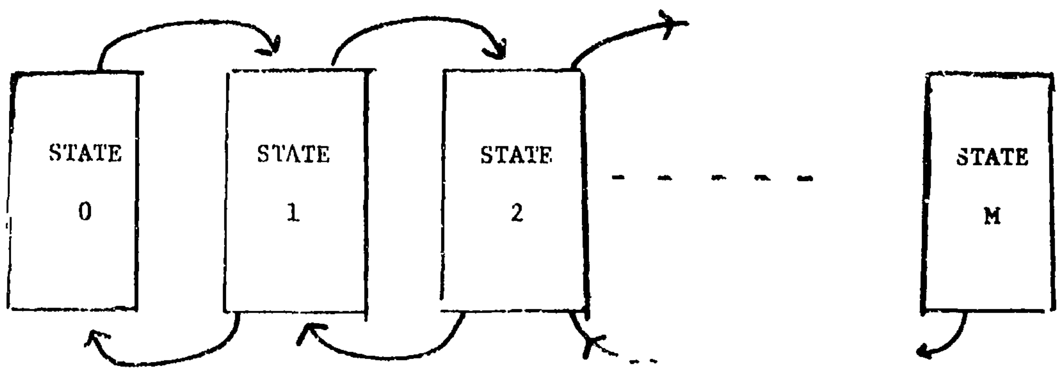
3.5.1 Full on redundancy

While the other approaches discussed to evaluate the reliability of redundant systems were classical, the following must be considered unique. It uses as a foundation the basics of a Markovian process but treats and combines these into an expectation model.

Given a system comprised of L operating redundant identical units (each with identical values of λ and μ, in addition all failures and repairs of units take on an exponential density as defined previously.) Assume that D units (D<L) as a minimum must be operating in order for the system to function. The system then has a number of possible operating states.

State	Defined as	
0	L units operating	
1	L-1 units operating	Satisfactory operating states.
2	L-2 units operating	
.	.	.
.	.	.
.	.	.
.	.	.
N	D units operating	
N + 1	D-1 units operating - a system failure	

As indicated previously transitions can take place only between adjacent states. That is, given the system is in state of j,(j>0), the system can next go to either state (j + 1) or (j - 1), no other. This follows from the flow chart,



Naturally, for example, when 2 of L units are failed the next transition must be either to 1 of L units failed (indicating a repair) or 3 of L units failed (indicating a failure).

It is also clear that immediately before the system fails (reaches state (N + 1)) the system must be in state (N).

Each state has associated with it unique failure rate λ_j, and repair rate μ_j, computed as:

$\lambda_j = (L-j)\lambda$ $\quad j \le N$ [N is the state number associated with D units operating]

$\mu_j = j\mu$ (assumes that as soon as a unit fails repair is begun).

Since we are dealing with exponential density functions for repair and failure:
Expected time in state j before transition

$$= E(j) = \frac{1}{\lambda_j + \mu_j} \tag{3.52}$$

Probability of going to state (j + 1) on the next transition, given you are in state j at the present time. (This, of course, signifies an additional failure)

$$P(j+1/j) = \frac{\lambda_j}{\lambda_j + \mu_j} \tag{3.53}$$

Probability of going to state (j-1) on the next transition, given you are in state j. (This of course signifies a repair)

$$= P(j-1/j) = \frac{\mu_j}{\lambda_j + \mu_j} \tag{3.54}$$

$$P(j-1/j) + P(j+1/j) = 1$$

Given the characteristics above and using expectation, the expected length of time to go from state j to state (j+1), E(j+1/j), can b formulated.

$$E(j+1/j) = P(j+1/j) \cdot E(j) + P(j-1/j) \left[E(j) + E(j/j-1) + E(j+1/j)\right]$$

Rearranging and grouping terms

$$E(j+1/j) = \frac{P(j+1/j) \cdot E(j) + P(j-1/j) \left[E(j) + E(j/j-1)\right]}{1 - P(j-1/j)}$$

Substituting (3.52) - (3.54) into the above:

$$E(j+1/j) = \frac{1}{\lambda_j} + \frac{\mu_j}{\lambda_j} E(j/j-1) = \frac{1}{\lambda_j} + \frac{\mu_j}{\lambda_j} \sum_{K=0}^{j-1} \frac{\prod_{n=K+1}^{j-1} \mu_n}{\prod_{n=K}^{j-1} \lambda_n} \tag{3.55}$$

It is obvious that in order to fail (assuming the units of the system were all operating at $t=0$) the system must gracefully degrade; that is, from state 1 it must eventually go to state 2, from state 2 it must eventually go to state 3, etc., and the average time for such graceful degradation from state j to (j+1) (taking into account transitions from j to (j-1)) is accounted for by (3.55) Therefore:

$$\sum_{j=0}^{N} E(j+1/j) = M \tag{3.56}$$

that is, the sum of the expected times to transition from state j to (j+1), from (j+1) to (j+2), etc., is the expected time to go from state 0 to state (N+1), which is the mean time to system failure. Repeated use of (3.55) in (3.56) leads to:

$$M = \sum_{j=0}^{N} E(j+1/j)$$

$$= \sum_{j=0}^{N} \frac{1}{\lambda_j} + \sum_{j=1}^{N} \sum_{K=0}^{j-1} \left(\frac{\prod_{n=K+1}^{j} \mu_n}{\prod_{n=K}^{j} \lambda_n} \right) \tag{3.57}$$

For example, take a two-unit redundant system as before (each unit having identical failure and repair rates λ, and μ) and apply (3.57). The result is:

$$M = \frac{\mu + 3\lambda}{2\lambda^2} \tag{3.58}$$

the same result as from (3.38) which evaluated the two unit system using a conventional Markovian Process.

As can be observed, this procedure is significantly simpler to apply than any of the others discussed and less time-consuming (due solely to the fact that application of (3.57) is all that is required). Its drawbacks are (1) while the mean time to failure of the system may be derived, its reliability expressed as the Probability of no system failure in time t cannot be determined; and (2) it is capable of handling only redundant systems comprised of units with identical failure and repair rates.

3.5.2 Stand-by Redundancy

The concepts of the previous section can be used to develop a relationship for mean time to failure for a system employing stand-by redundancy.

Given a system comprised of L redundant, identical units (each with identical values of λ, and μ. In addition, all failure and repairs of units take on an exponential density as defined previously). Assume that D units (D<L) must operate at all times in order for the system to operate. All other units are not energized and have $\lambda = 0$. They remain energized until a failure occurs and only then is one energized. A repair action is immediately started on the failed unit. The system fails when one of the D operating units fails and no energized unit is available to take its place (all (L-D) units under repair when a failure occurs). The system has a number of possible states:

State	Defined as
0	D Units operating, (L-D) energized, 0 under repair
1	D Units operating, (L-D-1) energized, 1 under repair
2	D Units operating, (L-D-2) energized, 2 under repair
•	• • • • • • • •
•	• • • • • • • •
•	• • • • • • • •
•	• • • • • • • •
N	D Units operating, 0 • (L-D) under repair
N+1	D-1 Units operating → Failure

Again, transition can take place only among adjacent states. That is, given the system is in state j (j>0), the system can next go to either state (j+1) or (j-1), no others. (See flowchart of last section).

Naturally, for example, when 2 of L units are undergoing repair (are failed) the next transition must be either to 1 of L units undergoing repairs (one unit repaired) or 3 of L units undergoing repairs (an other operating unit failed).

It is also clear that immediately before the system fails (reaches state (N+1)) the system must be in state (N).

Each state (j) has associated with it a failure rate of λ_j, and a repair rate of μ_j.

For stand-by redundancy

$\lambda_j = D\lambda$ $\quad$ $j \leq N$

$\mu_j = j\mu$ $\quad$ (assumes that as soon as a unit fails, repairs are begun).

Since we are dealing with exponential density functions for repairs and failure:
Expected time in state j before transition

$$= E(j) = \frac{1}{D\lambda+\mu_j} \tag{3.59}$$

Probability of going to state (j+1) on the next transition, given you are in state j at the present time. (Signifies an additional failure)

$$= P(j+1/j) = \frac{D\lambda}{D\lambda + \mu_j} \tag{3.60}$$

Probability of going to state (j-1) on the next transition, given you are in state j (signifies a repair)

$$= P(j-1/j) = \frac{\mu_j}{\lambda D + \mu_j} \tag{3.61}$$

$$P(j-1/j) + P(j+1/j) = 1$$

The expected length of time to go from state j to state (j+1), E(j+1/j), is formulated:

$$E(j+1/j) = P(j+1/j) \cdot E(j) + P(j-1/j)\left[E(j) + E(j/j-1) + E(j+1/j)\right]$$

Substituting (3.59) - (3.61) into the above, regrouping and simplifying.

$$E(j+1/j) = \frac{1}{D\lambda} + \frac{\mu_j}{D\lambda}\ E(j/j-1) \tag{3.62}$$

It is obvious that in order to fail (assuming the units of the system would all be in an operable condition at t= 0) the system must gradually degrade; that is, from say state 1 must eventually go to state 2, before going to state 3. And the average time for such a degradation from state j to state (j+1) (taking into account transitions from j to (j-1)) is accounted for by (3.62).

Therefore:

$$\sum_{j=0}^{N} E(j+1/j) = M \tag{3.63}$$

that is, the sum of the expected times to go from state j to j+1, from (j+1) to (j+2) etc. is the expected time to go from state 0 to state (N+1), the mean time to failure of the system.

Repeated use of (3.62) in (3.63) leads to:

$$M = \sum_{j=0}^{N} E(j+1/j) = \sum_{j=0}^{N} \frac{1}{\lambda D} + \sum_{j=1}^{N} \sum_{K=0}^{j-1} \frac{\prod_{n=K+1}^{j} \mu_n}{\prod_{n=K}^{j} D\lambda} \tag{3.64}$$

$$= \frac{(L-D+1)}{\lambda D} + \sum_{j=1}^{L-D} \sum_{K=0}^{j-1} \frac{\prod_{n=K+1}^{j} \mu_n}{(D\lambda)^{j-K+1}}$$

For example, take a two-unit redundant system in standby (each unit having identical failure and repair rates λ,μ) and apply (3.64). The result is:

$$M = \frac{2\lambda+\mu}{\lambda^2} \qquad (3.65)$$

the same as the result from (3.50) which directly evaluated the two unit standby system using a conventional Markovian Process.

As can be seen, this procedure, like the preceding, has advantages of simplicity and time over the classical method to evaluate the mean time to failure of a standby system. Its drawbacks are (1) while the mean time to failure of the system may be derived, its reliability expressed as the probability that the system will operate over an interval of time (0,T) with no failure cannot be determined; and (2) it is capable of handling only redundant systems comprised of units with identical failure and repair rates.

3.6 System Failure Rate Approach (For Full on Operations)

In the previous section, concepts pertaining to transition rates (failure and repair rates) between adjacent states were used as the foundation on which an evaluation technique was based. In this section we will discuss an evaluation technique based on the concept of System Failure Rate associated with each system state

Let us first define:

(1) System State - The description of the systems operating condition in terms of how many units are operating and how many are in a failed state.

The following is a list of the possible states for a parallel system comprised of L units.

States	Description of State
0	L Units operating, 0 failed
1	(L-1) Units operating, 1 failed
2	(L-2) Units operating, 2 failed
3	(L-3) Units operating, 3 failed
•	• • • • •
•	• • • • •
•	• • • • •
N	(L-N) Units operating, N failed
N+1	(L-N-1) Units operating, (N+1) failed

We bear in mind that transitions can take place only between adjacent states (more than one failure at one time has a probability = 0; more than one repair at one time has a probability = 0; a failure and a repair manifesting themselves simultaneously has a probability = 0). That is the system which if in state j (j>0) can go either to state (j+1) or state (j-1). Further, assuming that the system is in state 0, at t= 0, in order for the system to fail (reach a failed state, say (N+1)) it must at some time go from state 0 to state 1, from state 1 to state 2, from state 2 to state 3, etc, etc.

Further, if one defines:

E(j+1/j), the expected time to go from state j to state (j+1), then:

$$\sum_{j=0}^{N} E(j+1/j) = M = \text{Mean time to system failure.}$$

(2) System Substate - A Substate of a system state. Many system states as defined above (X units operating, Y failed) may occur in a different number of ways (for example, take a two-unit redundant system comprised of Unit A and Unit B. Unit A may be operating and Unit B failed, or Unit B may be operating and Unit A failed. Both are different substates belonging to the system state, one unit operating, one unit failed) each way in which the system state might occur is called a substate.

(3) Border State - A substate of a system state where the next unit failure will cause a system failure.

Examples:

(1) A three-unit redundant system, a minimum of any 2 units must be operating in order for the system to operate. In this case, the border states would be any 2 units operating, one unit failed. Three border states would result.

(2) A four-unit redundant system, a minimum of one unit must be operating in order for the system to operate. In this case a border state would be any one unit operating, 3 units failed. Four border states would result.

(4) Limiting Availability $= \dfrac{\mu}{\mu + \lambda} = \dfrac{M}{M + R}$

Defined and derived earlier as a non-parametric measure which indicates: the proportion of time that a unit is operating (or up), given its average failure and repair rates, λ and μ; or the probability that a unit is operating at a random point in time, given its average failure and repair rates λ and μ.

In a system comprised of L parallel units application of the Limiting Availability figure of merit can determine the proportion of the time that the system is operating or the proportion of time that the system is in a given state.

Example: A three-unit Redundant System made up of units A, B, and C, (A minimum of one unit is required for satisfactory operation).

Let A_0 indicate Unit A is on

A_F indicate Unit B is Failed

B_0 indicate Unit B is On

B_F indicate Unit B is Failed

C_0 indicate Unit C is On

C_F indicate Unit C is Failed

The system can be in any one of the following states:

$A_0B_0C_0$

$A_0B_0C_F$

$A_0B_FC_0$

$A_0B_FC_F$ Border State

$A_FB_0C_0$

$A_FB_0C_F$ Border State

$A_FB_FC_0$ Border State

$A_FB_FC_F$ Failed State

Let:

A_A, A_B, A_C represent the Availability of A, B and C respectively.

Let:

$(1-A_A)$, $(1-A_B)$, $(1-A_C)$ represent the Unavailability of A, B & C (This represents the proportion of time that A, B, or C is in a failed state (undergoing repair), or the probability that A, B or C is in a failed state (undergoing repair), at a random point in time).

Using the fundamentals of probability, we can now model each state and determine the proportion of time that a system is in any particular state. Taking the last example:

State	Proportion of time in state		Border state
$A_0\ B_0\ C_0$	$A_A A_B A_C$	$= \frac{\mu^3}{(\mu+\lambda)^3}$	No
$A_0\ B_0\ C_F$	$A_A A_B (1-A_C)$	$= \frac{\mu^2\lambda}{(\mu+\lambda)^3}$	No
$A_0\ B_F\ C_0$	$A_A(1-A_B)A_C$	$= \frac{\mu^2\lambda}{(\mu+\lambda)^3}$	No
$A_0\ B_F\ C_F$	$A_A(1-A_B)(1-A_C)$	$= \frac{\mu\lambda^2}{(\mu+\lambda)^3}$	Yes
$A_F\ B_0\ C_0$	$(1-A_A)A_B A_C$	$= \frac{\mu^2\lambda}{(\mu+\lambda)^3}$	No
$A_F\ B_0\ C_F$	$(1-A_A)A_B(1-A_C)$	$= \frac{\mu\lambda^2}{(\mu+\lambda)^3}$	Yes
$A_F\ B_F\ C_0$	$(1-A_A)(1-A_B)A_C$	$= \frac{\mu\lambda^2}{(\mu+\lambda)^3}$	Yes
$A_F\ B_F\ C_F$	$(1-A_A)(1-A_B)(1-A_C)$	$= \frac{\lambda^3}{(\mu+\lambda)^3}$	Failed
TOTAL=	1	1	

The primary concept to grasp in the application of this evaluation technique is the fact that the system has a failure rate equal to zero while it is in every state except a Border State; that is, the system can fail directly only from a Border State; it can not fail directly from any other state. The system failure rate λ_B, in a Border State is the sum of the failure rates of the operating units in that Border State.

$$\lambda_B = \sum_{i=1}^{D} \lambda_i$$

λ_B= failure rate of a Border State
λ_i= failure rate of the i^{th} operating unit (non failed unit) in the Border State.
D= Minimum number of units required for successful system operation.

If one were to associate with each Border State, the Product of the Proportion of time in that state (A_{Bi}) and the Sum of the Failure rates of the operating units in that state (λ_{Bi}) and an arbitrarily long system operational period T the (Note T includes the time that the system is operating satisfactorily and the time the system is down for repair) term:

$$A_{Bi}\ \lambda_{Bi}\ T$$

would represent the expected number of failures anticipated from Border state i over an arbitrarily long period of time T.

If one were to sum this result for all Border States, K, making up system state N (recalling system state N+1 is a failed state).

$$T\sum_{i=1}^{K} A_{Bi}\ \lambda_{Bi} = \text{the number of failures expected in System operational time T from Border States.} \tag{3.66}$$

(Recall failures can occur only from Border States, therefore all system failures occur from Border States).

If one were to sum the substate availabilities (A_{ji}) of all the states in which the system satisfactorily operates (including Border States) and multiply that sum by the same arbitrarily long period of time T

$$T\sum_{j=0}^{N}\sum_{i=1}^{Z_j} A_{ji} = \text{the total time that the system is operating satisfactorily.} \tag{3.67}$$

A_{ji} = The availability of substate i associated with system state j.
Z_j = Number of substates associated with system state j.

The ratio of (3.66) to (3.67) is:

$$\frac{\text{number of failures expected}}{\text{Total operating time}} \quad \text{or}$$

$$\frac{\sum_{j=0}^{N}\sum_{i=1}^{Z_j} A_{ji}}{\sum_{i=1}^{K} A_{Bi}\ \lambda_{Bi}} = \text{Average time to system failure measured steady state} = M_S \tag{3.68}$$

This measures the actual perceived average time to system failure over the life use of the system. It is important to note that since we are using state N as a minimum operating state, the model indicates that as soon as the system fails, it enters state (N+1). The system is then operational again as soon as repair puts it into State N. Each cycle of operation-failure for the system (after the first system failure) then starts in State N and ends in State (N+1).
The average time to failure (M_s) is in reality, the mean time to go from state N to state (N+1) or E(N+1/N).

Therefore, (3.68) may be rewritten as:

$$\frac{\sum_{j=0}^{N} \sum_{i=1}^{Z_j} A_{ji}}{\sum_{i=1}^{K} A_{Bi} \lambda_{Bi}} = E(N+1/N) = M_B \tag{3.69}$$

Taking a two-unit redundant system as an example (λ, and μ of both units identical) and applying (3.69)

$$E(N+1/N) = E(2/1) = \frac{\frac{\mu^2+2\lambda\mu}{(\mu+\lambda)^2}}{\frac{2\lambda^2\mu}{(\mu+\lambda)^2}} = \frac{\mu+2\lambda}{2\lambda^2} \tag{3.70}$$

which is identical to the result which would occur when equation (3.55) of the previous section is applied to evaluate F(N+1/N).

For the next step, define a new criteria for system failure follows: if a minimum of D operating units out of L were originally necessary for system operation, assume now that a minimum of (D+1) operating units out of L is required for system operation. Determine new values of A_{Bi}, λ_{Bi}, as before and apply equation (3.69) once more making N=(N-1). By changing the failure criteria from D to (D+1) the application of (3.69) really evaluates the expected time to go from state (N-1) to state N

E(N/N-1)

Repetition of the above a number of times until the boundary state shifts from N to 1 (see state chart at beginning of section) results in the following summation:

$$\sum_{j=0}^{N} E(j+1/j) = M =$$ Mean time to failure for the system assuming all units were operable at t= 0.

or:

$$T \sum_{j=N+1}^{M} \sum_{i=1}^{Z_j} A_{ji}$$

$$M = \sum_{v=0}^{N} \frac{\sum_{j=0}^{v} \sum_{i=1}^{Z_j} A_{ji}}{\sum_{i=1}^{K_v} A_{vi} \lambda_{vi}} \tag{3.71}$$

For example, take a two-unit system with identical values of λ and μ with operational scenario as before:

$$M = E(N+1/N) + E(N/N-1) = E(2/1) + E(1/0)$$

From (3.70), E (2/1) =

$$\frac{\mu+\lambda 2}{2\lambda^2}$$

$$E(1/0) = \frac{\frac{\mu^2}{(\mu+\lambda)^2}}{\frac{2\mu^2\lambda}{(\mu+\lambda)^2}} = \frac{1}{2\lambda} \tag{3.72}$$

$$M = \frac{\mu+2\lambda}{2\lambda^2} + \frac{1}{2\lambda} = \frac{\mu+3\lambda}{2\lambda^2}$$

which is identical to (3.58) which was derived using the expectation/transition approach.
In the event that all units have identical values of μ and λ (3.69) reduces to:

$$M_S = \frac{\sum_{i=0}^{N} \binom{L}{D+i} A_{s,(N-i)}}{\binom{\mu}{D} A_B \lambda_B} \tag{3.73}$$

A_{sj} = Availability of any substate in state j
A_{sj} = A_s,(i+1), A_s, (i+2), ...A_B
A_B = Availability of any border state (all border states have same availability if all units identical).
In the event that all units have identical values of λ and μ (3.71) reduces to:

$$M = \sum_{v=0}^{N} \frac{\sum_{i=0}^{N-v} \binom{L}{D+v+i} A_{s,(N-v-i)}}{\binom{L}{D+v} \left(A_{(N-v)}\right)\left(\lambda_{(N-v)}\right)} \tag{3.74}$$

$A_{(N-v)}$ = Availability of any substate in system state (N-v)

$$\lambda_{(N-v)} = \sum_{i=1}^{D+v} \lambda_i$$

λ_i = failure rate of the i^{th} operating unit in state $(N-v)$.

For the two-unit redundant example previously described:

$$M_s = \frac{2\lambda+\mu}{2\lambda^2}$$

$$M = \frac{\mu+3\lambda}{2\lambda^2}$$

which of course duplicates the results of (3.72)

As can be seen, this procedure like the preceding, has the advantages of simplification and time over the classical methods to evaluate the mean time to failure of a full-on system. It has one advantage over the previous expectation transition, combination method in that it is capable of handling systems of parallel units of different values of λ and μ. Its shortcoming is that the reliability of the system expressed as the probability that the system will operate over an interval of time (0,T) with no failure, cannot be determined.

3.7 Systems Periodically Maintained

In previous sections, we have considered systems which were not maintained, and systems which were maintained immediately after failure. In this section, we will consider redundant systems which are only periodically maintained (a system is placed in operation, then left unattended; every T hours a maintenance team visits the system and repairs all unit failures). Let

$f(t)$ = density function of failure for the redundant system.

Then

$R(T) = 1 - \int_0^T f(t)dt$ = probability that the system will be on at the end of T.

If the system is still operating at T, then the operating time for system is T. If the system fails at t in (0,T), then the operating time for system is t. Therefore, the average uninterrupted operating time of a system in (0,T), M_T, is given by

$$M_T = TR(T) + \int_0^T tf(t)dt$$

$$= TR(T) + t(1-R(t))\Big|_0^T - \int_0^T (1-R(t))dt$$

$$= \int_0^T R(t)dt$$

It is possible for a system to fail before the first cycle (0,T) is complete or it is possible that the system will not fail until the Nth cycle is complete. Therefore, if we had a large number of such systems (X) in the field and intended to so maintain these over a long period:

$R(T)$ = proportion of systems surviving the first cycle with no failure

$R(T)^2$ = proportion of systems surviving 2 cycles with no failure

$R(T)^3$ = proportion of systems surviving 3 cycles with no failure

•

•

•

$R(T)^N$ Proportion of system surviving the first N cycles with no failure. Therefore, of the original X systems

Cycle No.

1 X systems would operate uninterruptably for an average of M_T hours each before failure

2 R(T)X systems would operate uninterruptably for an additional M_T hours.

3 $R(T)^2X$ systems would operate uninterruptably for an additional M_T hours.

•

•

•

N $R(T)^{N-1}$ X systems would operate uninterruptably for an additional M_T hours.

And the average uninterrupted operating time to first failure per system is

$$= \frac{XM_T + \sum_{i=1}^{N-1} \left[R(T)\right]^i X M_T}{X} = M_T\left[1 + \sum_{i=1}^{N-1} \left[R(T)\right]^i\right]$$

But

$$1 + \sum_{i=1}^{N-1} \left[R(T)\right]^i = \text{a progression of the form}$$

$$a,\ ar,\ ar^2 \ldots . ar^n$$

with sum:

$$S_N = \frac{\left[R(T)\right]^N - 1}{R(T) - 1}$$

Since:

$$\left[R(T)\right]^N < 1 \qquad S_N \to \frac{1}{1-R(T)} \quad \text{as N gets arbitrarily}$$

large. Hence:
Average uninterrupted operating time to first failure M_{FF}

$$M_{FF} = \frac{M_T}{1-R(T)} = \frac{\int_0^T R(t)dt}{1-R(T)}$$

3.8 Impact of Redundancy on Maintainabilily

The analysis of redundant systems is almost always concerned with the impact of that decision on reliability or mean time to failure. It is seldom related to impact on maintainability and mean time to repair $\bar{R}$ and total maintenance hours required (which impacts support cost). Yet, redundancy impacts these areas critically. The following sections quantitatively describe such effects.

3.8.1 Redundancy Impact on Mean Time to Repair (Full on Redundancy)

Take a system composed of L units, D of which have to operate in order for the system to function satisfactorily. Recalling the notation of the last section define those states in which the system is considered failed. This would be states:

N+ 1 When (D-1) units are operating, (L-D+1) under repair
N + 2 When (D-2) units are operating, (L-D+2) under repair
• • • •
• • • •
• • • •
L When 0 units are operating, L under repair.

Define the substates associated with each state (Z_j) and the availability of each substate A_{ji} (where A_{ji} = the availability of the ith substate associated with each system state j) Through knowledge of the availability components of such states and substates, we will form the ratio:

$$\left[\frac{\text{Expected time the system is in a failed state}}{\text{Expected number of transitions from an operational state to a failed state (Failures)}}\right]$$

$$= \text{the expected time that the system is in failed state} \qquad (3.75)$$

From (3.67), the number of failures expected in T:

$$= T \sum_{i=1}^{K} A_{Bi} \quad \lambda_{Bi}$$

$$\therefore \frac{\sum_{j=N+1}^{L} \sum_{i=1}^{Z_j} A_{ji}}{\sum_{i=1}^{K} A_{Bi} \quad \lambda_{Bi}} = \text{average system down time} \qquad (3.76)$$

when all units have identical values of λ, and μ the average system down time

$$= \frac{\sum_{i=1}^{D} \binom{L}{D-i} A_{s,(N+i)}}{\binom{L}{D} A_B \lambda_B}$$

For example, for a 2 unit parallel system with equal values of λ and μ, the average system down time

$$= \frac{\dfrac{\lambda^2}{(\mu+\lambda)^2}}{\dfrac{2\lambda^2\mu}{(\mu+\lambda)^2}} = \frac{1}{2\mu} = \frac{\bar{R}}{2}$$

3.8.2 Redundancy effect on Total Maintenance Time (Full on Redundancy)

Let:

$(1-A_i)$= Proportion of time unit is under repair (assumes repair starts as soon as unit fails).

$T(1-A_i)$= Expected time unit is under repair (over a long time interval 0,T).

Given L redundant units

$$T \sum_{i=1}^{L} (1-A_i) = \text{Total maintenance time expended on system} \qquad (3.77)$$

If all units have identical values of λ, and μ (3.77) reduces to

$$\frac{\lambda TL}{\mu + \lambda} = \text{Total maintenance time expended on system} \qquad (3.78)$$

Comparing the above with a simplex system with total maintenance time expenditure of:

$$\frac{\lambda T}{\mu+\lambda}$$

It is clear that with every unit added to increase reliability maintenance time increases proportionately.

3.9 Efficient levels of Redundancy

The question arises as to what degree redundancy should be applied.

Should it better be applied at the system level, subsystem level or the unit level? There is, of course, no general answer to this question. Much depends on the nature of the application at hand. In some instances, due to practicality, cost or the engineering nature of the system itself, only one course of action is possible. In the event, however, that no constraints are evident on the level of redundancy, which level should be chosen?

Assume that the system in question is denoted as A in Figure 3.3. A may be partitioned at will into (L) modules, all having identical failure rates λ, and the total failure rate of $A=L\lambda$. It is necessary to improve the reliability of A and the only available means to realize this improvement is through the application of redundancy. How does the level of redundancy chosen affect reliability?

Let us assume that due to reasons of economy only one redundant unit can be considered. Shall we

(1) make system A redundant with System B (Its redundant entity)?

(2) break up System A into L modules? break up system B into L modules and make each module of B redundant to its corresponding A module? And, if we choose the latter, what is the sensitivity of reliability to the partitioning scheme chosen?

Using (2.3) the mean time to first failure (M) of the system described equal to

$$M=\frac{1}{\lambda}\sum_{K=1}^{L}(-1)^{K+1}\frac{L!}{K!(L-K)!}\sum_{S=1}^{2K}\frac{1}{S}$$

Since in this case N= 2

where: L= number of modules the system can be broken down to

λ= failure rate of each module

$\lambda_0=L\lambda$= failure rate of the single system

$\frac{1}{\lambda_0}=\frac{1}{L\lambda}$= mean time between failure of the single system.

The above equation may be written as:

$$M=\frac{L}{\lambda_0}\sum_{K=1}^{L}(-1)^{K+1}\frac{L!}{K!(L-K)!}\sum_{S=1}^{2K}\frac{1}{S}$$

Treating L in the above as a variable and λ_0 as a constant, M can be evaluated as a function of the degree of partitioning practiced (the value of L).

Assuming that for each module, formed connectors and perhaps even buffers or transducers must be added such that the failure rate increases as the number of modules increases,

$$M = \frac{L}{\lambda_0 + p'(\lambda_0)} \sum_{R=1}^{L} (-1)^{K+1} \frac{L!}{K!\,(L-K)!} \sum_{S=1}^{2K} \frac{1}{S}$$

where p= proportionate increase in single module failure rate as a consequence of redundancy application.

For p= 0 and p= .1

M is plotted in Figure (3.4) As a function of (λM).

END

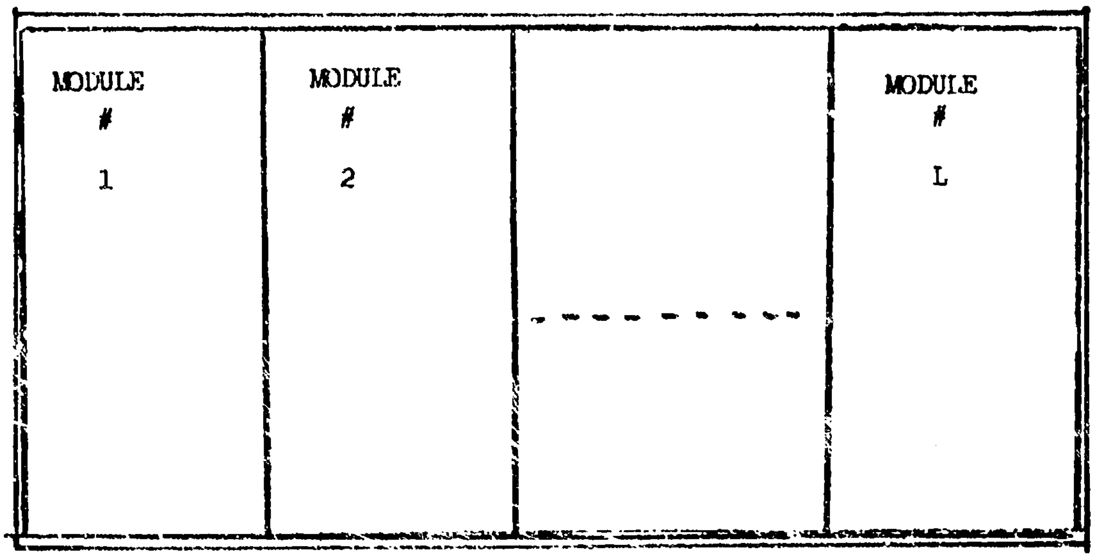
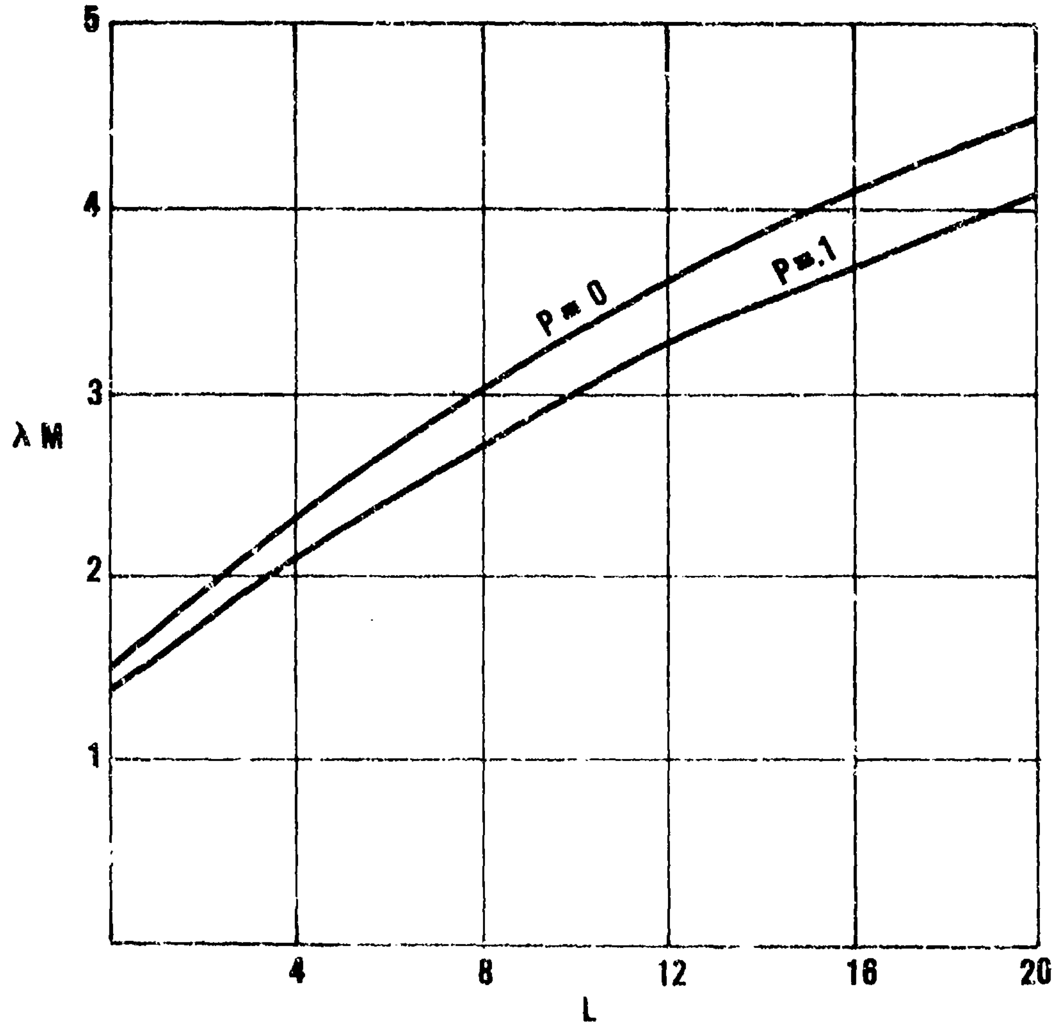


Figure 3.3 Division of Equipment A into Any Number of Modules



L = # of Modules the equipment is partitioned into
λ = Failure rate/Module

Reliability Improvement As A Consequence of Partitioning
Figure 3.4

APPENDIX A

From (2.2) we have:

$$M = \int_0^{\infty} \left[1 - (1 - e^{-\lambda t})^N\right]^L dt \qquad \text{(A1)}$$

Let:

$$p = e^{-\lambda t}$$

Then (A1) is transformed into:

$$\frac{1}{\lambda} \int_0^1 \frac{\left[1 - (1-p)^N\right]^L}{p} dp$$

Recalling the special case of the binomial series:

$$\frac{1}{\lambda} \int_0^1 \frac{\left[1 - (1-p)^N\right]^L}{p} dp \qquad \text{(A2)}$$

and equating $(1-p)^N = x$, (A2) is transformed into:

$$\frac{1}{\lambda} \int_0^1 \frac{1}{p} \sum_{k=0}^{L} \binom{L}{k} (-1)^k (1-p)^{kN} dp$$

$$= \frac{1}{\lambda} \int_0^1 \frac{1}{p} dp + \frac{1}{\lambda} \sum_{k=1}^{L} \binom{L}{k} (-1)^k \int_0^1 \frac{(1-p)^{kN}}{p} dp \qquad \text{(A3)}$$

Now integrating by parts yields:

$$\int_0^1 \frac{(1-p)^{kN}}{p} dp = - \sum_{s=1}^{kN} \frac{1}{s} + \int_0^1 \frac{1}{p} dp$$

(A3) can therefore be expressed as:

$$M = \frac{1}{\lambda} \sum_{k=1}^{L} \binom{L}{k} (-1)^{k+1} \sum_{s=1}^{kN} \frac{1}{s}$$

which implies (2.3)

APPENDIX B

Taking Laplace transforms of equations (3.18) to (3.20) and defining initial conditions yields:

$$1 = (2\lambda+s)\, p_2(s) - \mu p_1(s)$$

$$0 = -(s+\lambda+\mu)\, p_1(s) + 2\lambda p_2(s) + 2\mu p_0(s)$$

$$0 = \lambda p_1(s) - (s+2\mu)\, p_0(\ )$$

the above simultaneous equations can be easily solved for $p_0(s)$

$$p_0(s) = \frac{2\lambda^2}{s(s+\mu+\lambda)\left[s+2(\mu+\lambda)\right]}$$

which implies

$$p_0(t) = \frac{\lambda^2}{(\lambda+\mu)^2} + \frac{\lambda^2}{(\lambda+\mu)^2}\, e^{-2(\lambda+\mu)t} - \frac{2\lambda^2 e^{-(\lambda+\mu)t}}{(\lambda+\mu)^2}$$

Noting that

$$p(t) = p_2(t) + p_3(t) = 1 - p_0(t)$$

we have (3.25).

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